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A braided Yang–Baxter algebra in a theory of two coupled lattice quantum KdV: algebraic properties and ABA representations

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Abstract

A generalization of the Yang–Baxter algebra is found in quantizing the monodromy matrix of two (m)KdV equations discretized on a space lattice. This braided Yang–Baxter equation still ensures that the transfer matrix generates operators in involution which form the Cartan sub-algebra of the braided quantum group. Representations diagonalizing these operators are described through relying on an easy generalization of algebraic Bethe ansatz techniques. The conjecture that this monodromy matrix algebra leads, *in the cylinder continuum limit*, to a perturbed minimal conformal field theory description is analysed and supported.

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1. Introduction

After Liouville, an integrable system (with infinite degrees of freedom) is usually defined to be a (1+1)-dimensional classical (or quantum) field theory with the property of having an infinite number of *integrals of motion in involution* (IMI). Among them one may be chosen and called the Hamiltonian (operator). As for quantum systems the IMI do not help the determination of the most intriguing and interesting features of these systems because of their Abelian character. However, one can single out at least two different starting points to overcome this difficulty and both make use of non-Abelian algebras and only partially of the Abelian one.

One starting point leaves from the classical theory of integrable systems and more specifically from the Lax pair formulation of non-linear partial differential equations [1]. Usually, the Poisson structure of the Lax zero-curvature formulation is encoded in a classical

r-matrix [2–4] which assures the integrability by entering the Poisson *classical Yang–Baxter algebra* for the entries of the *monodromy matrix*. However, a classical Yang–Baxter algebra is the expression of an algebraic structure deeper than the Abelian one [5]. Indeed, at the quantum level a classical Yang–Baxter algebra becomes a (quantum) Yang–Baxter algebra ([6, 7] and references therein) which is nothing other than a definition relation of a quantum group, a deformation of a usual Lie algebra [8–10]. As for looking at the representations of the quantum group from the viewpoint of the spectrum of the Hamiltonian operator, a very efficient evolution of the Bethe ansatz—the algebraic Bethe ansatz (ABA)—has been founded initially for the sine–Gordon field theory [11] and then developed for many models ([7] and references therein). In other words, an infinite dimensional non-Abelian algebra *includes* the Abelian algebra and allows us to build the spectrum of the Hamiltonian operator (and of the other IMI) as its representation in terms of operators on a Hilbert space (sometimes the Hermitian norm on the space is possibly negative, though always non-degenerate). More recently, it has been possible to write down exact non-linear equations describing the energy spectrum of (twisted) sine–Gordon field theory on a cylinder [12, 13].

Another starting point is based on statistical field theory and, in particular, on the very important fact that fixed points of the renormalization group are described by conformal field theories (CFTs), i.e. theories where the correlation functions are covariant under the conformal group [14]. In 2D the conformal algebra is infinite dimensional (the Gelfand–Fuks–Virasoro algebra [15]) and the 2D-CFTs are simple integrable quantum theories enjoying as their own crucial property the covariance under an infinite dimensional Virasoro symmetry [16]. As for the integrability *à la Liouville* the CFT possesses a bigger $\mathcal{W}$-like symmetry and, in particular, it is invariant under different infinite dimensional Abelian sub-algebras of the latter [17]. Each of these Abelian sub-algebras is generated by the IMI, which can be constructed in terms of the Virasoro algebra—the real new ingredient in these theories since it is a true field and state spectrum generating symmetry. Indeed, the Verma modules over this algebra turn out to be reducible because of the occurrence of sub-modules generated over the so-called *singular vectors* [18]. The factor-module by the maximal proper sub-module can be endowed with a non-degenerate Hermitian Shapovalov form and the singular vectors are characterized to produce a null Hermitian product with all the other vectors. Now, this factor-module is isomorphic to the Hilbert space of the local fields (or states) in 2D-CFTs and its own properties lead to a number of very interesting algebraic–geometrical features such as character expressions, fusion algebras, differential equations for correlation functions, etc (see [19] for a review). Unfortunately, this beautiful picture collapses when one pushes the system away from criticality by perturbing the original CFT with some relevant local field: from the infinite dimensional Virasoro symmetry only the finite dimensional Poincaré sub-algebra survives the perturbation. After suitable deformations, at least a conformal Abelian sub-algebra survives the perturbation, resulting in the off-critical Abelian algebra [20]. As said before, this symmetry does not carry sufficient information to find the energy spectrum by means of IMI alone, but it constitutes a very useful help in determining other interesting quantities. For instance, scattering theory corresponding to off-critical theories is usually well known and contains solitons (or kinks), anti-solitons (or anti-kinks) and a number of bound states. The mass spectrum and the S-matrix of different integrable field theories have been known for about a few years [21]. Despite this on-shell information, the off-shell quantum field theory is much less developed. In particular, the computation of the corresponding correlation functions is still an important open problem. Actually, some progress in this direction has been made, since the exact form-factors (FFs) of several local fields were computed (see, for instance, [22, 23]). This allows one to make predictions about the long-distance behaviour of the corresponding correlation functions. On the other hand, some efforts have been made to estimate the short-distance behaviour of the theory in the context of the so-called conformal perturbation theory (CPT) [23]. By combining the previous techniques (FFs and CPT), it has been possible to estimate several interesting physical quantities ([24] and references therein). In addition and in the direction of determining in an approximative way the first energy levels of the simplest perturbed minimal conformal field theories on a cylinder, very good results have been obtained by the truncated conformal space technique, developed in [25]. From those results the plane geometry can be recovered as the limit of cylinder size goes to infinity, conditional on having a good numerical estimate for large size, which is not so easily obtained.

Consequently, one important problem in perturbed conformal field theories (PCFTs), i.e. theories formulated following the second starting point, is the exact construction of the spectrum of the Hamiltonian operator—and possibly of the other IMI—in the more general situation of the cylinder geometry, by using the idea of the first approach (ABA). This *synergetic* combination of both the previous approaches is difficult in many cases, i.e. in all the cases where a Lax formulation of the classical version of the off-critical theory is missing. Actually, even a quantum Lax formulation of CFTs is only partially presented and disentangled in the literature [26–28].

Among the huge variety of integrable theories of the aforementioned kind, the prototype is the very interesting case of minimal conformal field theories [16] perturbed by the $\Phi_{1,3}$ primary operator [20]. In this paper, a (regularized) lattice integrable definition of the quantum Lax operator will be given both for the CFT and for the off-critical theory. Besides, a deep analysis of its algebraic and integrable properties will be carried out to disentangle the algebraic structure behind the integrability of the monodromy matrix and of the transfer matrix: a generalization of the Yang–Baxter equation will be found. In conclusion, a suitable modification of the ABA will be applied to determine the eigenvalues and eigenstates of the lattice transfer matrix, the *generating function* of all the IMI. Actually, all the other integrable perturbations of minimal conformal field theories would be exhausted by treating analogously the conformal case described in [28], but we will leave this completion for a forthcoming paper [29].

In section 2, we present a brief introduction to classical ($A_1^{(1)}$ modified) KdV theory from the point of view of Lax pair and CFT. In particular, we show how the space discretization of the monodromy matrix arises in a very natural way. In section 3, we look at CFT as quantization of the KdV theory and then propose two left and right lattice regularized quantum Lax operators. We also calculate explicitly the exchange relations satisfied by these Lax operators on different sites. In section 4, we give a general theorem about the exchange relation satisfied by a general succession of left and right Lax operators: the conclusion is that in any case we end up with a braided Yang–Baxter algebra, still ensuring Liouville integrability. In addition, we single out two *conformal* monodromy matrices and two *off-critical* monodromy matrices. In section 5, we set up the first step towards the generalization of the algebraic Bethe ansatz method to braided Yang–Baxter algebras: the coordinate representation of the basic entries of the lattice Lax operator. In section 6, we perform the generalized ABA in the case of conformal monodromy matrices finding explicitly Bethe equations and transfer matrix eigenvalues/eigenvectors. We argue about the insights that these monodromy matrices describe in the continuum limit the chiral and anti-chiral part of the minimal CFTs on a cylinder. In section 7, we perform the ABA in the case of off-critical monodromy matrices finding explicitly Bethe equations and transfer matrix eigenvalues/eigenvectors. In section 8, we analyse the *conformal* limit on the off-critical transfer matrices eigenvalues. In section 9, we disentangle the structure of the critical and off-critical monodromy matrices in the operatorial scaling limit to gain understanding about the physical meaning of these theories: in the off-critical case we guess again that they are equivalent monodromy matrix descriptions of minimal CFTs perturbed by the $\Phi_{1,3}$ operator. In section 10, we find a connection between our braided ABA results and those of the usual ABA in lattice sine–Gordon theory (LSGT). In section 11, we summarize our results and give hints about future investigations.

2. An introduction to the ($A_1^{(1)}$ modified) KdV theory

It is well known from [17, 26] that the conformal field theory symmetry algebra,

$$U(y) = -\frac{c}{24} + \sum_{-\infty}^{+\infty} L_{-n}\,\mathrm{e}^{\mathrm{i}ny} \tag{2.1}$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n} \tag{2.2}$$

becomes the second Poisson structure of the usual KdV hierarchy [30],

$$\{u(y), u(z)\} = 2[u(y) + u(z)]\delta'(y-z) + \delta'''(y-z) \tag{2.3}$$

in *the classical limit* (central charge $c \to -\infty$), provided the substitutions

$$U(y) \to -\frac{c}{6}u(y) \qquad [*, *'] \to \frac{6\pi}{\mathrm{i}c}\{*, *'\} \tag{2.4}$$

are performed. Besides, it has also been established by Drinfeld and Sokolov [30] how generalized *modified* KdV hierarchies are built through the centreless Kac–Moody algebras and how the generalized KdV hierarchies correspond to inequivalent nodes of the Dynkin diagram. In the case of $A_1^{(1)}$ Dynkin diagram we have the usual KdV hierarchy. For quantization reasons, we shall start from the usual *modified* KdV equation

$$\partial_\tau v = \tfrac{3}{2}v^2 v' + \tfrac{1}{4}v''' \tag{2.5}$$

which describes the temporal flow for the spatial derivative $v = -\varphi'$ of a Darboux field defined on a spatial interval $y \in [0, R]$, recalling the connection to the KdV variable $u(y)$ through the Miura transformation [31]:

$$u(y) = \varphi'(y)^2 - \mathrm{i}\varphi''(y). \tag{2.6}$$

Assuming quasi-periodic boundary conditions on φ, it verifies by definition the Poisson bracket

$$\{\varphi(y), \varphi(y')\} = -\frac{1}{2}s\left(\frac{y-y'}{R}\right) \tag{2.7}$$

where $s(z)$ is the quasi-periodic extension of the sign function

$$s(z) = 2n+1 \quad n < z < n+1 \qquad s(n) = 2n \quad n \in \mathbb{Z}. \tag{2.8}$$

As a consequence, the mKdV variable $v(y)$ satisfies a non-ultralocal Poisson bracket

$$\{v(y), v(y')\} = \frac{\partial}{\partial y}\delta^{(p)}(y-y') \tag{2.9}$$

the non-ultralocality being expressed by the derivative of the R-periodic delta function $\delta^{(p)}(y)$. Besides, this Poisson structure implies the second Poisson structure to the KdV field u (2.3), which is still non-ultralocal.

Now, equation (2.5) can be rewritten as a null curvature condition:

$$[\partial_\tau - \mathrm{l}', \partial_y - \mathrm{l}] = 0 \tag{2.10}$$

for connections belonging to the $A_1^{(1)}$ loop algebra:

$$\mathrm{l} = -\mathrm{i}vh + (e_0 + e_1) \tag{2.11}$$

$$\mathrm{l}' = \lambda^2(e_0 + e_1 - \mathrm{i}vh) + \frac{1}{2}\left[(v^2 + \mathrm{i}v')e_0 + (v^2 - \mathrm{i}v')e_1\right] - \frac{1}{2}\left(\mathrm{i}\frac{v''}{2} + \mathrm{i}v^3\right)h \tag{2.12}$$

where the generators e_0, e_1, h are chosen in the canonical gradation of the loop algebra, i.e.

$$e_0 = \lambda E \qquad e_1 = \lambda F \qquad h = H \tag{2.13}$$

with E, F, H generators of the A_1 Lie algebra:

$$[H, E] = 2E \qquad [H, F] = -2F \qquad [E, F] = H. \tag{2.14}$$

For simplicity we deal with the fundamental representation of A_1:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.15}$$

A remarkable geometrical interest is attached to the monodromy matrix which realizes the parallel transport along the *space* and which is the solution of the boundary value problem:

$$\partial_y m(y; \lambda) = l(y; \lambda) m(y; \lambda) \qquad m(0; \lambda) = \mathbf{1}. \tag{2.16}$$

After indicating with $\mathcal{P}$ the path-order product, the formal solution of the previous equation,

$$m(y; \lambda) = \mathcal{P} \exp\left(\int_0^y \mathrm{d}y'\, l(y', \lambda)\right) \tag{2.17}$$

allows us to calculate the equal time Poisson brackets between the entries of the monodromy matrix

$$m(\lambda) \equiv m(R; \lambda) = \mathcal{P} \exp\left(\int_0^R \mathrm{d}y\, l(y, \lambda)\right) \tag{2.18}$$

provided those of the connection l are known. The result is that the Poisson brackets between the entries of the monodromy matrix are fixed by the so-called classical r-matrix in the (classical) Yang–Baxter Poisson bracket equation:

$$\{m(\lambda) \overset{\otimes}{,} m(\lambda')\} = [r(\lambda/\lambda'), m(\lambda) \otimes m(\lambda')]. \tag{2.19}$$

In our particular case the r-matrix is the trigonometric one:

$$r(\lambda) = \frac{\lambda + \lambda^{-1}}{\lambda - \lambda^{-1}} \frac{H \otimes H}{2} + \frac{2}{\lambda - \lambda^{-1}}(E \otimes F + F \otimes E). \tag{2.20}$$

By carrying through the trace on both members of the Poisson brackets (2.19), we are allowed to conclude that the transfer matrix

$$t(\lambda) = \mathrm{Tr}\, m(\lambda) \tag{2.21}$$

Poisson commutes with itself for different values of the spectral parameter:

$$\{t(\lambda), t(\lambda')\} = 0. \tag{2.22}$$

From this relation, we can say that $t(\lambda)$ is the generating function of the classical IMI by expanding it, for instance, in powers of λ. As an important example, we obtain a series of local IMI I_{2n-1}^{cl} from the asymptotic expansion,

$$\lambda \to \infty \qquad \frac{1}{2\pi} \ln t(\lambda) \asymp \lambda - \sum_{n=1}^{\infty} c_n \lambda^{(1-2n)} I_{2n-1}^{cl} \tag{2.23}$$

where c_n are real coefficients (see, e.g., [27] for their expression). Property (2.22) guarantees the integrability of the model *à la* Liouville and all the local IMI are expressed in terms of u: for instance the first ones are

$$I_1^{cl} = -\frac{1}{2}\int_0^R \mathrm{d}y\, u(y) \qquad I_3^{cl} = -\frac{1}{8}\int_0^R \mathrm{d}y\, u^2(y). \tag{2.24}$$

The equation of motion corresponding to the choice of I_3^{cl} as Hamiltonian

$$\partial_\tau v = \left\{ I_3^{cl}, v \right\} \tag{2.25}$$

is the mKdV equation (2.5) itself.

In addition, we can introduce *the right version* of the mKdV equation:

$$\partial_{\bar\tau} \bar v = \tfrac{3}{2}\bar v^2 \bar v' + \tfrac{1}{4}\bar v''' \tag{2.26}$$

where $\bar v = -\bar\varphi'$ and the right quasi-periodic Darboux variable, $\bar\varphi(\bar y)$, $0 \leqslant \bar y \leqslant R$, satisfies the Poisson bracket (with a change of sign):

$$\{\bar\varphi(\bar y), \bar\varphi(\bar y')\} = \frac{1}{2} s\left(\frac{\bar y - \bar y'}{R}\right) \tag{2.27}$$

and Poisson commutes with the left variable $\varphi(y)$. Equation (2.26) derives as in the left case from a null curvature condition for right connections $\bar l$ and $\bar l'$, whose espressions are given by formulae (2.11), (2.12) where v has been replaced by $\bar v$.

Formulae for monodromy and transfer matrices are also analogous to the left case:

$$\bar m(\lambda) = \mathcal{P}\exp\left(\int_0^R \mathrm{d}\bar y\; \bar l(\bar y, \lambda)\right) \qquad \bar t(\lambda) = \mathrm{Tr}\, \bar m(\lambda). \tag{2.28}$$

The Poisson bracket between the entries of the monodromy matrix differs by a sign from the left counterpart, which still implies the Poisson-commutativity for the right transfer matrix. Hence $\bar t(\lambda)$ generates in its asymptotic expansion the right classical local IMI:

$$\lambda \to \infty \qquad \frac{1}{2\pi}\ln \bar t(\lambda) \asymp \lambda - \sum_{n=1}^{\infty} c_n \lambda^{(1-2n)} \bar I_{2n-1}^{cl} \tag{2.29}$$

where the $\bar I_{2n-1}^{cl}$ are given by the expressions for I_{2n-1}^{cl} where φ has been replaced by $\bar\varphi$.

Owing to the opposite sign in (2.27), the right mKdV equation is obtained through the right action of $\bar I_3$:

$$\partial_{\bar\tau} \bar v = \left\{ \bar v, \bar I_3^{cl} \right\}. \tag{2.30}$$

A very natural way to quantize a classical theory, in the presence of path-ordering and avoiding the problems of ultraviolet divergences, is to put it on the lattice and then to quantize the discretized theory. Of course, in case of an integrable theory the integrability (expressed in our case by the classical Yang–Baxter equation (2.19) and then by the quantum *braided* Yang–Baxter equation) has to be preserved by discretization and quantization.

Hence, let us divide the interval [0, R] into $2N$ parts and define the discretized Darboux variables:

$$\varphi_k \equiv \varphi(y_k) \qquad \bar\varphi_k \equiv \bar\varphi(\bar y_k) \qquad y_k \equiv \bar y_k \equiv k\frac{R}{2N} \qquad k \in \mathbb{Z}. \tag{2.31}$$

As a consequence of (2.7), (2.27) they satisfy

$$\{\varphi_k, \varphi_h\} = -\frac{1}{2} s\left(\frac{k-h}{2N}\right) \qquad \{\bar\varphi_k, \bar\varphi_h\} = \frac{1}{2} s\left(\frac{k-h}{2N}\right) \qquad \{\varphi_k, \bar\varphi_h\} = 0. \tag{2.32}$$

We define again for $m \in \mathbb{Z}$

$$v_m^- \equiv \tfrac{1}{2}[(\varphi_{2m-1} - \varphi_{2m+1}) + (\varphi_{2m-2} - \varphi_{2m}) - (\bar{\varphi}_{2m-1} - \bar{\varphi}_{2m+1}) + (\bar{\varphi}_{2m-2} - \bar{\varphi}_{2m})] \tag{2.33}$$

$$v_m^+ \equiv \tfrac{1}{2}[(\bar{\varphi}_{2m-1} - \bar{\varphi}_{2m+1}) + (\bar{\varphi}_{2m-2} - \bar{\varphi}_{2m}) - (\varphi_{2m-1} - \varphi_{2m+1}) + (\varphi_{2m-2} - \varphi_{2m})]. \tag{2.34}$$

Note that the fields $v_m^\pm$ are periodic, i.e. $v_{m+N}^\pm = v_m^\pm$. As a consequence, we can confine ourselves to the fields $v_m^\pm$ with $1 \leqslant m \leqslant N$. Note also that the fields $v_m^\pm$ live on a lattice which has half the number of sites of the lattice on which φ_k and $\bar{\varphi}_k$ live. We will indicate with

$$\Delta = \frac{R}{N} \tag{2.35}$$

the lattice spacing of the $v_m^\pm$ lattice.

Because of (2.32) the operators $v_m^\pm$ enjoy the following non-ultralocal Poisson brackets:

$$\{v_m^+, v_n^+\} = \tfrac{1}{2}\left(\delta_{m-1,n}^{(p)} - \delta_{m,n-1}^{(p)}\right) \tag{2.36}$$

$$\{v_m^-, v_n^-\} = -\tfrac{1}{2}\left(\delta_{m-1,n}^{(p)} - \delta_{m,n-1}^{(p)}\right) \tag{2.37}$$

$$\{v_m^+, v_n^-\} = -\tfrac{1}{2}\left(\delta_{m-1,n}^{(p)} - 2\delta_{m,n}^{(p)} + \delta_{m,n-1}^{(p)}\right) \tag{2.38}$$

where the N-periodic Kronecker delta is defined by

$$\delta_{m,n}^{(p)} \equiv \begin{cases} 1 & \text{if} \quad (m-n) \in N\mathbb{Z} \\ 0 & \text{otherwise.} \end{cases} \tag{2.39}$$

Therefore, introducing

$$w_m^\pm = \mathrm{e}^{\mathrm{i}v_m^\pm} \tag{2.40}$$

we define the discrete left and right Lax matrices, respectively,

$$l_m(\lambda) = \begin{pmatrix} (w_m^-)^{-1} & \Delta\lambda w_m^+ \\ \Delta\lambda (w_m^+)^{-1} & w_m^- \end{pmatrix} \qquad \bar{l}_m(\lambda) = \begin{pmatrix} (w_m^+)^{-1} & \Delta\lambda w_m^- \\ \Delta\lambda (w_m^-)^{-1} & w_m^+ \end{pmatrix} \tag{2.41}$$

in terms of which the discretized versions of monodromy matrices (2.18) and (2.28) are

$$m(\lambda) = l_N(\lambda) l_{N-1}(\lambda) \ldots l_2(\lambda) l_1(\lambda) \tag{2.42}$$

$$\bar{m}(\lambda) = \bar{l}_N(\lambda) \bar{l}_{N-1}(\lambda) \ldots \bar{l}_2(\lambda) \bar{l}_1(\lambda). \tag{2.43}$$

Indeed, in the cylinder limit defined by

$$N \to \infty \quad \text{and fixed} \quad R \equiv N\Delta \tag{2.44}$$

we obtain the scaling equalities

$$v_m^- = -\Delta\varphi'(y_{2m}) + O(\Delta^2) \qquad v_m^+ = -\Delta\bar{\varphi}'(\bar{y}_{2m}) + O(\Delta^2) \tag{2.45}$$

from which

$$l_m(\lambda) = 1 + \Delta l\left(m\frac{R}{N}, \lambda\right) + O(\Delta^2) \qquad \bar{l}_m(\lambda) = 1 + \Delta\bar{l}\left(m\frac{R}{N}, \lambda\right) + O(\Delta^2). \tag{2.46}$$

Therefore, the discretized monodromy matrices in the scaling limit behave as follows:

$$m(\lambda) = \prod_{k=1}^{N}\left[1 + \Delta l\left(k\frac{R}{N}, \lambda\right) + O(\Delta^2)\right] \to \mathcal{P}\exp\int_0^R \mathrm{d}y\, l(y, \lambda) = m(\lambda)$$

$$\bar{m}(\lambda) = \prod_{k=1}^{N}\left[1 + \Delta\bar{l}\left(k\frac{R}{N}, \lambda\right) + O(\Delta^2)\right] \to \mathcal{P}\exp\int_0^R \mathrm{d}\bar{y}\, \bar{l}(\bar{y}, \lambda) = \bar{m}(\lambda)$$

i.e. they reproduce the monodromy matrices for the left and right KdV theory.

In the following section, we will quantize the discretized monodromy matrices (2.42), (2.43) in order to build quantum versions of the left and right KdV theories.

3. Quantum version of the KdV theory

The quantum counterparts of the classical local IMI in the KdV theory are local IMI in conformal field theories [17] (after suitable deformation they are local IMI in minimal CFTs perturbed by the operator $\Phi_{1,3}$ [20, 32]). They are constructed in terms of the quantizations of the Darboux fields, the Feigin–Fuks left and right bosons [18], which we will indicate with $\phi(y)$ and $\bar{\phi}(y)$. They are defined to be operators quasi-periodic in y and $\bar{y}$ verifying the *canonical* (light-cone) commutation relations:

$$[\phi(y),\phi(y')] = -\frac{\mathrm{i}\pi\beta^2}{2}s\left(\frac{y-y'}{R}\right) \qquad [\bar{\phi}(\bar{y}),\bar{\phi}(\bar{y}')] = \frac{\mathrm{i}\pi\beta^2}{2}s\left(\frac{\bar{y}-\bar{y}'}{R}\right) \tag{3.1}$$

where β^2 is a real positive constant, and commuting with each other. By virtue of quasi-periodicity, the fields ϕ and $\bar{\phi}$ can be expanded in modes as follows:

$$\phi(y) = Q + \frac{2\pi y}{R}P - \mathrm{i}\sum_{n\neq 0}\frac{a_{-n}}{n}\,\mathrm{e}^{\mathrm{i}\frac{2\pi}{R}ny} \tag{3.2}$$

$$\bar{\phi}(\bar{y}) = \bar{Q} - \frac{2\pi\bar{y}}{R}\bar{P} - \mathrm{i}\sum_{n\neq 0}\frac{\bar{a}_{-n}}{n}\,\mathrm{e}^{-\mathrm{i}\frac{2\pi}{R}n\bar{y}} \tag{3.3}$$

and the commutation relations (3.1) impose that the left and right modes form two commuting Heisenberg algebras:

$$[Q,P] = [\bar{Q},\bar{P}] = \frac{\mathrm{i}}{2}\beta^2 \qquad [a_n,a_m] = [\bar{a}_n,\bar{a}_m] = \frac{n}{2}\beta^2\delta_{n+m,0} \tag{3.4}$$

acting respectively on the left and right space whose tensor product defines the vector space of a conformal field theory (sometimes the Hermitian norm on the space is possibly negative, though always non-degenerate). In this way, the operators ϕ realize a free field representation of the Virasoro algebra according to the quantum version of the Miura transformation, called Feigin–Fuks construction [18]:

$$U(y) = \beta^{-2} : \phi'(y)^2 : + \mathrm{i}(1-\beta^{-2})\phi''(y) - \tfrac{1}{24} \tag{3.5}$$

where the symbol normal ordering :: means, as usual, that modes with bigger index n must be placed to the right. The central charge of this representation of the Virasoro algebra is

$$c = 13 - 6(\beta^2 + \beta^{-2}). \tag{3.6}$$

A whole hierarchy of commuting quantities is built using density polynomials of powers of $U(y)$ and its derivatives and they constitute the chiral quantum local IMI of CFTs [17]:

$$I_{2k-1} = \int_0^R \mathrm{d}y\; U_{2k}(y). \tag{3.7}$$

For example, the first densities are

$$U_2(y) = -\tfrac{1}{2}U(y) \qquad U_4(y) = -\tfrac{1}{8} : U^2(y) :. \tag{3.8}$$

Of course, after replacing ϕ with $\bar{\phi}$, the same construction holds for the right theory. We can define a right Virasoro algebra (we assume the same central charge as the left algebra)

$$\bar{U}(\bar{y}) = \beta^{-2} : \bar{\phi}'(\bar{y})^2 : + \mathrm{i}(1-\beta^{-2})\bar{\phi}''(\bar{y}) - \tfrac{1}{24} \tag{3.9}$$

in terms of which a right hierarchy of commuting quantities is defined according to formulae (3.7) and (3.8), by replacing U with $\bar{U}$. They constitute the right local IMI of CFTs.

In the classical limit (2.4) $\beta \to 0$ and hence

$$[*,*'] \to \mathrm{i}\pi\beta^2\{*,*'\} \qquad U(y) \to \beta^{-2}u(y) \qquad \bar{U}(\bar{y}) \to \beta^{-2}\bar{u}(\bar{y}) \tag{3.10}$$

in such a way that (3.5), (3.9) become the Miura transformations and the IMI of conformal field theories reduce to the IMI of the KdV theory. Of course, the quantum Feigin–Fuks operators ϕ, $\bar{\phi}$ reduce to the classical Darboux fields φ, $\bar{\varphi}$, respectively.

In a natural way we have approached the problem of defining the quantum versions of the monodromy matrices (2.18), (2.28), so that we are in the position of deriving expressions for the transfer matrices and their eigenvectors and eigenvalues. This corresponds to finding and diagonalizing the local IMI and also the non-local IMI [4, 27, 28, 33] of quantum KdV (and these IMI are part of those of CFT [28]). Besides, we note that the continuum methodology developed in a series of beautiful papers by Bazhanov *et al* [27] uses slightly different monodromy matrices than those to which ours reduce in the cylinder scaling limit (2.44). However, we want to remain faithful to the usual definition of monodromy matrix even in the non-ultralocal case: we will leave the analysis of the connections [27] to another paper [29]. Besides, the construction of a lattice theory will allow us to get rid of ultraviolet divergence problems (this statement is pretty obvious but it will be proved in the next paper [29]) and to use the algebraic Bethe ansatz techniques to diagonalize the transfer matrix. For all these reasons our starting point is the quantization of the classical discretized monodromy matrices (2.42), (2.43).

Let us start with the left case. The discretized quantum Feigin–Fuks bosons $\phi_k, \bar{\phi}_k$, $k \in \mathbb{Z}$, satisfy (see (2.32), (3.10))

$$[\phi_k, \phi_h] = -\frac{\mathrm{i}\pi\beta^2}{2} s\left(\frac{k-h}{2N}\right) \qquad [\bar{\phi}_k, \bar{\phi}_h] = \frac{\mathrm{i}\pi\beta^2}{2} s\left(\frac{k-h}{2N}\right) \qquad [\phi_k, \bar{\phi}_h] = 0. \quad (3.11)$$

We define the lattice variables $V_m^{\pm}$, $m \in \mathbb{Z}$, as quantizations of the classical ones, $v_m^{\pm}$ (2.33), (2.34):

$$V_m^- \equiv \tfrac{1}{2}[(\phi_{2m-1} - \phi_{2m+1}) + (\phi_{2m-2} - \phi_{2m}) - (\bar{\phi}_{2m-1} - \bar{\phi}_{2m+1}) + (\bar{\phi}_{2m-2} - \bar{\phi}_{2m})] \quad (3.12)$$

$$V_m^+ \equiv \tfrac{1}{2}[(\bar{\phi}_{2m-1} - \bar{\phi}_{2m+1}) + (\bar{\phi}_{2m-2} - \bar{\phi}_{2m}) - (\phi_{2m-1} - \phi_{2m+1}) + (\phi_{2m-2} - \phi_{2m})]. \quad (3.13)$$

They are periodic discrete variables: $V_m^{\pm} = V_{m+N}^{\pm}$. Hence, without loss of generality, we may again restrict ourselves to consider only $V_m^{\pm}$ with $1 \leqslant m \leqslant N$. These variables satisfy the non-ultralocal commutation relations ($1 \leqslant m, n \leqslant N$):

$$[V_m^+, V_n^+] = \frac{\mathrm{i}\pi\beta^2}{2}\left(\delta_{m-1,n}^{(p)} - \delta_{m,n-1}^{(p)}\right) \quad (3.14)$$

$$[V_m^-, V_n^-] = -\frac{\mathrm{i}\pi\beta^2}{2}\left(\delta_{m-1,n}^{(p)} - \delta_{m,n-1}^{(p)}\right) \quad (3.15)$$

$$[V_m^+, V_n^-] = -\frac{\mathrm{i}\pi\beta^2}{2}\left(\delta_{m-1,n}^{(p)} - 2\delta_{m,n}^{(p)} + \delta_{m,n-1}^{(p)}\right). \quad (3.16)$$

Therefore, after defining

$$W_m^{\pm} \equiv \mathrm{e}^{\mathrm{i}V_m^{\pm}} \qquad q \equiv \mathrm{e}^{-\mathrm{i}\pi\beta^2} \quad (3.17)$$

we can derive from the commutator algebra (3.14)–(3.16) the exchange algebra

$$\begin{aligned} &W_{m+1}^{\pm} W_m^{\pm} = q^{\pm\frac{1}{2}} W_m^{\pm} W_{m+1}^{\pm} \qquad W_{m+1}^{\pm} W_m^{\mp} = q^{\mp\frac{1}{2}} W_m^{\mp} W_{m+1}^{\pm} \\ &W_m^+ W_m^- = q W_m^- W_m^+ \\ &[W_m^{\sharp}, W_n^{\sharp'}] = 0 \qquad \text{if} \quad (1 \leqslant n \leqslant N) \quad 2 \leqslant |m-n| \leqslant N-2 \end{aligned} \quad (3.18)$$

with the obvious identification $W_{N+1}^{\pm} = W_1^{\pm}$ and with $\sharp$ and $\sharp'$ both equal to + or −. Plus or minus part of this algebra has been introduced in [34]. The whole algebra was considered in [35], however, without being constructed in terms of Feigin–Fuks bosons. At the end, we present the discrete Lax operators

$$L_m(\lambda) \equiv \begin{pmatrix} (W_m^-)^{-1} & \Delta\lambda W_m^+ \\ \Delta\lambda (W_m^+)^{-1} & W_m^- \end{pmatrix} \qquad \bar{L}_m(\lambda) \equiv \begin{pmatrix} (W_m^+)^{-1} & \Delta\lambda W_m^- \\ \Delta\lambda (W_m^-)^{-1} & W_m^+ \end{pmatrix} \tag{3.19}$$

which are a quantization of the discrete left and right Lax matrices (2.41). Operators $\bar{L}_m$ and L_m are connected by a *duality* transformation, which exchanges W_m^+ with W_m^- or equivalently q with q^{-1} in relations (3.18).

Although, our interest is in introducing the operators L_m starting from the Feigin–Fuks bosons, they were already introduced in [35] for defining the discretized monodromy matrix of the (left) mKdV theory as

$$L_N(\lambda)L_{N-1}(\lambda)\ldots L_2(\lambda)L_1(\lambda). \tag{3.20}$$

As will be clear in the following, this definition is perfectly correct, although the ABA solution of the problem in [35] contains an *ab initio* mistake which affects the final results (the author of [35] is in agreement with our finding [36]). Moreover, we will give a meaning and a role to the right counterpart of L_m, the Lax operator $\bar{L}_m$.

Now, it is important for the following to derive the exchange relations for left and right Lax operators (3.19). Hence, let us consider the quantum R-matrix and the quantum Z-matrix, the matrix encoding the braiding, as introduced in [35] for the left mKdV problem (however we remark that in the Z-matrix given in [35] q should be replaced by $q^{1/2}$):

$$R_{ab}(\xi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\xi^{-1}-\xi}{q^{-1}\xi^{-1}-q\xi} & \frac{q^{-1}-q}{q^{-1}\xi^{-1}-q\xi} & 0 \\ 0 & \frac{q^{-1}-q}{q^{-1}\xi^{-1}-q\xi} & \frac{\xi^{-1}-\xi}{q^{-1}\xi^{-1}-q\xi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.21}$$

$$Z_{ab} = \begin{pmatrix} q^{-\frac{1}{2}} & 0 & 0 & 0 \\ 0 & q^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & q^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & q^{-\frac{1}{2}} \end{pmatrix}. \tag{3.22}$$

R_{ab} and Z_{ab} act on the tensor product $a \otimes b$ of two auxiliary two-dimensional spaces. Using only exchange relations (3.18) one can extend the results [35] for the left case showing that the operators (3.19) satisfy the following relations ($1 \leqslant m \leqslant N$):

$$R_{ab}\left(\frac{\lambda}{\lambda'}\right) L_{am}(\lambda)L_{bm}(\lambda') = L_{bm}(\lambda')L_{am}(\lambda)R_{ab}\left(\frac{\lambda}{\lambda'}\right) \tag{3.23}$$

$$R_{ab}\left(\frac{\lambda'}{\lambda}\right) \bar{L}_{am}(\lambda)\bar{L}_{bm}(\lambda') = \bar{L}_{bm}(\lambda')\bar{L}_{am}(\lambda)R_{ab}\left(\frac{\lambda'}{\lambda}\right) \tag{3.24}$$

$$L_{am}(\lambda)L_{bm+1}(\lambda') = L_{bm+1}(\lambda')Z_{ab}^{-1}L_{am}(\lambda) \tag{3.25}$$

$$\bar{L}_{am}(\lambda)\bar{L}_{bm+1}(\lambda') = \bar{L}_{bm+1}(\lambda')Z_{ab}\bar{L}_{am}(\lambda) \tag{3.26}$$

$$L_{am}(\lambda)\bar{L}_{bm+1}(\lambda') = \bar{L}_{bm+1}(\lambda')Z_{ab}^{-1}L_{am}(\lambda) \tag{3.27}$$

$$\bar{L}_{am}(\lambda)L_{bm+1}(\lambda') = L_{bm+1}(\lambda')Z_{ab}\bar{L}_{am}(\lambda). \tag{3.28}$$

In these equations we have defined $L_{am} \equiv L_m(\lambda) \otimes \mathbf{1}$ and $L_{bm} \equiv \mathbf{1} \otimes L_m(\lambda)$ and identified operators on the site $N+1$ with operators on the site 1. The first two relations are just Yang–Baxter equations, while the others describe the non-ultralocality, i.e. the fact that Lax operators on first-neighbouring sites and different auxiliary spaces do not commute. Of course, operators (3.19) on different auxiliary spaces and on sites m and n commute if $2 \leqslant |m-n| \leqslant N-2$.

In spite of this complication, it has been shown in [35] that the monodromy matrix (3.20) satisfies a modified version of the Yang–Baxter equation, called the braided Yang–Baxter equation, and that the corresponding transfer matrices are commuting operators for different values of the spectral parameter.

4. Braided Yang–Baxter algebra and integrals of motion

In this section we will define in a general way monodromy matrices as products of operators L and $\bar{L}$ (3.19) in every possible order. Then we will prove that every monodromy matrix generates the braided Yang–Baxter algebra.

Let us introduce the following site operators ($1 \leqslant m \leqslant N$):

$$K_m(\lambda) \equiv \chi_m L_m(\lambda\delta_m) + \bar{\chi}_m \bar{L}_m\left(\frac{\delta_m}{\lambda}\right) \tag{4.1}$$

where, for a fixed m, the real numbers χ_m, $\bar{\chi}_m$ may take only the two sets of values,

$$\{\chi_m = 0, \bar{\chi}_m = 1\} \qquad \text{or} \qquad \{\chi_m = 1, \bar{\chi}_m = 0\} \tag{4.2}$$

whereas δ_m are arbitrary complex parameters. In other words, on a fixed lattice site m, the operator $K_m(\lambda)$ can be equal to $L_m(\lambda\delta_m)$ or $\bar{L}_m(\delta_m/\lambda)$.

Now we are in a position to define in complete generality the monodromy matrix mentioned at the beginning of this section:

$$\Pi(\lambda) \equiv K_N(\lambda)\ldots K_1(\lambda). \tag{4.3}$$

Thanks to (4.1), (4.2) the matrix (4.3) is an ordered product of operators which for a fixed lattice site m may be equal to $L_m(\lambda\delta_m)$ or $\bar{L}_m(\delta_m/\lambda)$. In particular, the left monodromy matrix (3.20) of [35] is obtained when $\chi_m = 1, \delta_m = 1, \ \forall m$. Besides, the right analogue of this monodromy matrix is obtained when $\bar{\chi}_m = 1, \delta_m = 1, \ \forall m$.

Let us now state the key theorem of this section.

Theorem 1. *The monodromy matrix (4.3) satisfies for $N \geqslant 2$ the following braided relations:*

$$R_{ab}\left(\frac{\lambda}{\lambda'}\right)\Pi_a(\lambda)\left[\chi_N Z_{ab}^{-1} + \bar{\chi}_N Z_{ab}\right]\Pi_b(\lambda') = \Pi_b(\lambda')\left[\chi_N Z_{ab}^{-1} + \bar{\chi}_N Z_{ab}\right]\Pi_a(\lambda) R_{ab}\left(\frac{\lambda}{\lambda'}\right). \tag{4.4}$$

Proof. The proof follows by the repeated applications of relations (3.23)–(3.28). □

Definition 1. *An associative algebra generated by the entries $\Pi_{ij}(\lambda)$ of a 2×2 matrix $\Pi(\lambda)$ satisfying the relation*

$$R_{ab}\left(\frac{\lambda}{\lambda'}\right)\mathcal{Z}_{ba}^{-1}\Pi_a(\lambda)\hat{\mathcal{Z}}_{ab}^{-1}\Pi_b(\lambda') = \mathcal{Z}_{ab}^{-1}\Pi_b(\lambda')\hat{\mathcal{Z}}_{ba}^{-1}\Pi_a(\lambda)R_{ab}\left(\frac{\lambda}{\lambda'}\right) \tag{4.5}$$

where $R_{ab}(\xi)$, $\mathcal{Z}_{ab}$ and $\hat{\mathcal{Z}}_{ab}$ are 4×4 numerical matrices obeying

$$R_{ab}(\xi)R_{ac}(\xi\xi')R_{bc}(\xi') = R_{bc}(\xi')R_{ac}(\xi\xi')R_{ab}(\xi) \tag{4.6}$$

$$\mathcal{Z}_{ab}\mathcal{Z}_{ac}\mathcal{Z}_{bc} = \mathcal{Z}_{bc}\mathcal{Z}_{ac}\mathcal{Z}_{ab} \tag{4.7}$$

$$\hat{\mathcal{Z}}_{ab}\hat{\mathcal{Z}}_{ac}\mathcal{Z}_{bc} = \mathcal{Z}_{bc}\hat{\mathcal{Z}}_{ac}\hat{\mathcal{Z}}_{ab} \tag{4.8}$$

$$R_{ba}(\xi)\hat{\mathcal{Z}}_{ac}\hat{\mathcal{Z}}_{bc} = \hat{\mathcal{Z}}_{bc}\hat{\mathcal{Z}}_{ac}R_{ba}(\xi) \tag{4.9}$$

$$R_{cb}(\xi)\hat{\mathcal{Z}}_{ac}\hat{\mathcal{Z}}_{ab} = \hat{\mathcal{Z}}_{ab}\hat{\mathcal{Z}}_{ac}R_{cb}(\xi) \tag{4.10}$$

$$R_{ba}(\xi)\mathcal{Z}_{ac}\mathcal{Z}_{bc} = \mathcal{Z}_{bc}\mathcal{Z}_{ac}R_{ba}(\xi) \tag{4.11}$$

$$R_{cb}(\xi)\mathcal{Z}_{ac}\mathcal{Z}_{ab} = \mathcal{Z}_{ab}\mathcal{Z}_{ac}R_{cb}(\xi) \tag{4.12}$$

is called the braided Yang–Baxter algebra. *Equation (4.5) is called* the braided Yang–Baxter equation.

Braided Yang–Baxter algebras have been introduced in [37].

Equations (4.6)–(4.12) guarantee the associativity of the triple product:

$$\Pi_a(\lambda)\hat{\mathcal{Z}}_{ab}^{-1}\Pi_b(\lambda')\hat{\mathcal{Z}}_{ac}^{-1}\hat{\mathcal{Z}}_{bc}^{-1}\Pi_c(\lambda''). \tag{4.13}$$

In our specific case R_{ab} is given by (3.21), while

$$\mathcal{Z}_{ab} = \hat{\mathcal{Z}}_{ab} = \left[\chi_N Z_{ab} + \bar{\chi}_N Z_{ab}^{-1}\right]. \tag{4.14}$$

Since

$$[R_{ab}(\xi), Z_{ab}] = 0 \tag{4.15}$$

relation (4.5) reduces to (4.4).

Matrix (3.21) is well known to satisfy Yang–Baxter equation (4.6) and from (4.15) and the fact that Z_{ab} is diagonal the other associativity conditions (4.7)–(4.12) follow straightforwardly.

The braided Yang–Baxter algebra is a generalization of the usual Yang–Baxter algebra in the sense that in the particular case $\mathcal{Z}_{ab} = \hat{\mathcal{Z}}_{ab} = \mathbf{1}$ the former reduces to the latter. In our particular case, after looking at the form of Z_{ab} (3.22), we can say that this may occur only for the special value of the deforming parameter $q = 1$: this is why we call this algebra a braided generalization of Yang–Baxter algebra rather than a deformed generalization.

We also observe that a simple consequence of theorem 1 is that there is no way to reproduce the Yang–Baxter algebra by *fusing* site Lax operators (4.1): therefore the presence of the braided Yang–Baxter equation is an unavoidable feature of our approach, which, in its turn, leaves very naturally from the algebraic formulation of CFTs.

As a corollary of the previous theorem, we now prove the Liouville integrability.

Corollary 1. *The* λ*-dependent transfer matrix*

$$\sigma(\lambda) \equiv \mathrm{Tr}\,\Pi(\lambda) \tag{4.16}$$

commutes with itself at different values of λ:

$$[\mathrm{Tr}\,\Pi(\lambda), \mathrm{Tr}\,\Pi(\lambda')] = 0. \tag{4.17}$$

Proof. After multiplying relation (4.4) by $\chi_N Z_{ab} + \bar{\chi}_N Z_{ab}^{-1}$ and using the aforementioned property,

$$[R_{ab}(\lambda), Z_{ab}] = 0 \tag{4.18}$$

we obtain

$$\begin{aligned}\left[\chi_N Z_{ab} + \bar{\chi}_N Z_{ab}^{-1}\right]\Pi_a(\lambda)\left[\chi_N Z_{ab}^{-1} + \bar{\chi}_N Z_{ab}\right]\Pi_b(\lambda') \\ = R_{ab}\left(\frac{\lambda}{\lambda'}\right)^{-1}\left[\chi_N Z_{ab} + \bar{\chi}_N Z_{ab}^{-1}\right]\Pi_b(\lambda')\left[\chi_N Z_{ab}^{-1} + \bar{\chi}_N Z_{ab}\right]\Pi_a(\lambda)R_{ab}\left(\frac{\lambda}{\lambda'}\right).\end{aligned}$$

Then, from the cyclicity of the trace, we have

$$\begin{aligned}\mathrm{Tr}_{ab}\left\{\left[\chi_N Z_{ab} + \bar{\chi}_N Z_{ab}^{-1}\right]\Pi_a(\lambda)\left[\chi_N Z_{ab}^{-1} + \bar{\chi}_N Z_{ab}\right]\Pi_b(\lambda')\right\} \\ = \mathrm{Tr}_{ab}\left\{\left[\chi_N Z_{ab} + \bar{\chi}_N Z_{ab}^{-1}\right]\Pi_b(\lambda')\left[\chi_N Z_{ab}^{-1} + \bar{\chi}_N Z_{ab}\right]\Pi_a(\lambda)\right\}.\end{aligned} \tag{4.19}$$

From the diagonal structure of Z we can write explicitly

$$\left[\chi_N Z + \bar{\chi}_N Z^{-1}\right]_{\alpha_1\alpha_2}^{\beta_1\beta_2} = z_{\alpha_1\alpha_2}\delta_{\alpha_1}^{\beta_1}\delta_{\alpha_2}^{\beta_2} \qquad \left[\chi_N Z^{-1} + \bar{\chi}_N Z\right]_{\alpha_1\alpha_2}^{\beta_1\beta_2} = z_{\alpha_1\alpha_2}^{-1}\delta_{\alpha_1}^{\beta_1}\delta_{\alpha_2}^{\beta_2} \tag{4.20}$$

where $z_{\alpha_1\alpha_2}$ are some complex numbers. Hence, property (4.19) can be rewritten explicitly as

$$\sum_{\alpha_1,\alpha_2} z_{\alpha_1\alpha_2}\Pi(\lambda)_{\alpha_1}^{\alpha_1} z_{\alpha_1\alpha_2}^{-1}\Pi(\lambda')_{\alpha_2}^{\alpha_2} = \sum_{\alpha_1,\alpha_2} z_{\alpha_1\alpha_2}\Pi(\lambda')_{\alpha_2}^{\alpha_2} z_{\alpha_1\alpha_2}^{-1}\Pi(\lambda)_{\alpha_1}^{\alpha_1} \tag{4.21}$$

which shows the commutativity of the transfer matrices $\mathrm{Tr}\,\Pi(\lambda)$ for different values of the spectral parameter λ. □

At the end of this section, we define some important examples of monodromy matrices which we will deal with.

- Conformal case:
  1. Left monodromy matrix
  $$\chi_m = 1 \quad \bar{\chi}_m = 0 \quad \delta_m = 1 \quad \Rightarrow \quad \Pi(\lambda) = M(\lambda) \equiv L_N(\lambda)\ldots L_1(\lambda); \tag{4.22}$$
  2. Right monodromy matrix
  $$\chi_m = 0 \quad \bar{\chi}_m = 1 \quad \delta_m = 1 \quad \Rightarrow \quad \Pi(\lambda) = \bar{M}(\lambda) \equiv \bar{L}_N\left(\frac{1}{\lambda}\right)\ldots\bar{L}_1\left(\frac{1}{\lambda}\right). \tag{4.23}$$
- Off-critical case:
  1. Case right–left (r–l)
  $$\chi_{4i} = \chi_{4i-1} = 0 \qquad \bar{\chi}_{4i-2} = \bar{\chi}_{4i-3} = 0 \qquad \left(1 \leqslant i \leqslant \frac{N}{4},\ N \in 4\mathbb{N}\right)$$
  $$\delta_m = \mu^{\frac{1}{2}} \quad (1 \leqslant m \leqslant N)$$
  $$\Rightarrow \quad \Pi(\lambda) = \mathbf{M}(\lambda) \equiv \bar{L}_N\left(\frac{\mu^{\frac{1}{2}}}{\lambda}\right)\bar{L}_{N-1}\left(\frac{\mu^{\frac{1}{2}}}{\lambda}\right)L_{N-2}\left(\lambda\mu^{\frac{1}{2}}\right)L_{N-3}\left(\lambda\mu^{\frac{1}{2}}\right)\ldots\bar{L}_4$$
  $$\times\left(\frac{\mu^{\frac{1}{2}}}{\lambda}\right)\bar{L}_3\left(\frac{\mu^{\frac{1}{2}}}{\lambda}\right)L_2\left(\lambda\mu^{\frac{1}{2}}\right)L_1\left(\lambda\mu^{\frac{1}{2}}\right); \tag{4.24}$$
  2. Case left–right (l–r)
  $$\bar{\chi}_{4i} = \bar{\chi}_{4i-1} = 0 \qquad \chi_{4i-2} = \chi_{4i-3} = 0 \qquad \left(1 \leqslant i \leqslant \frac{N}{4}, N \in 4\mathbb{N}\right)$$
  $$\delta_m = \mu^{\frac{1}{2}} \quad (1 \leqslant m \leqslant N)$$
  $$\Rightarrow \quad \Pi(\lambda) = \mathbf{M}'(\lambda) \equiv L_N\left(\lambda\mu^{\frac{1}{2}}\right)L_{N-1}\left(\lambda\mu^{\frac{1}{2}}\right)\bar{L}_{N-2}\left(\frac{\mu^{\frac{1}{2}}}{\lambda}\right)\bar{L}_{N-3}\left(\frac{\mu^{\frac{1}{2}}}{\lambda}\right)\cdots$$
  $$\times L_4\left(\lambda\mu^{\frac{1}{2}}\right)L_3\left(\lambda\mu^{\frac{1}{2}}\right)\bar{L}_2\left(\frac{\mu^{\frac{1}{2}}}{\lambda}\right)\bar{L}_1\left(\frac{\mu^{\frac{1}{2}}}{\lambda}\right). \tag{4.25}$$

Now, we must give some explanation about names which have a physical origin. The monodromy matrix (4.22) has been introduced as a natural discretized version of that describing quantum KdV theory, i.e. the left part of CFT [27]. The monodromy matrix (4.23) is simply its right counterpart, completing the description of CFT. The quantum KdV description of CFT exhibits, for particular values of β^2, the usual features of conformal minimal CFTs perturbed by the $\Phi_{1,3}$ operator (e.g. the form of local IMI) [28]. Hence, this formulation should be very suitable for going into the off-critical region preserving integrability and our proposal (4.24), (4.25) for the description of CFTs perturbed by the $\Phi_{1,3}$ operator is now very natural. In any case, we will bring other supports to our conjecture in the following by diagonalizing the transfer matrices corresponding to (4.22)–(4.25) through ABA techniques.

5. Coordinate representation

In order to settle down a suitable generalization of ABA to the braided Yang–Baxter equation, it is useful to rewrite the Lax operators (3.19) in a coordinate representation.

Let us first recall the *position–momentum* Heisenberg algebra, generated by elements x_m, p_m, $1 \leqslant m \leqslant N$, satisfying

$$[x_m, x_n] = 0 \qquad [p_m, p_n] = 0 \qquad [x_m, p_n] = \frac{\mathrm{i}\pi\beta^2}{2}\delta_{m,n}. \tag{5.1}$$

The key observation is that we can realize the quantum generators $V_m^{\pm}$ for $1 \leqslant m \leqslant N$ (3.12), (3.13) by using *position* and *momentum* x_m, p_m:

$$V_m^{\pm} = \pm(x_{m+1} - x_m) - p_m \tag{5.2}$$

where the algebra element x_{N+1} is identified with x_1 or, although unnecessary for the following, we may think of x_h and p_h ($h \in \mathbb{Z}$) as N-periodic objects in h. In any case, it is easy to verify that elements (5.2) satisfy commutation relations (3.14)–(3.16).

Now, we may use the usual coordinate representation $\hat{x}_m, \hat{p}_m$ for the elements x_m, p_m, respectively, [11] to obtain a coordinate representation for $V_m^{\pm}$.

Let us indicate by $\mathcal{H}$ the *enlarged* vector space consisting of the $L^2(\mathbb{R})$ functions and of the distributions. Let us consider the N-tensor product of $\mathcal{H}$, $\mathcal{T}(\mathcal{H}) = \mathcal{H} \otimes \cdots \otimes \mathcal{H}$. The representative operators for the positions $\hat{x}_m$, $1 \leqslant m \leqslant N$, act multiplicatively on the vectors of $\mathcal{T}(\mathcal{H})$:

$$(\hat{x}_m\psi)(x_1, \ldots, x_N) = x_m\psi(x_1, \ldots, x_N) \tag{5.3}$$

while the representative operators for the momenta $\hat{p}_m$ act as derivations:

$$(\hat{p}_m\psi)(x_1, \ldots, x_N) = -\frac{\mathrm{i}\pi\beta^2}{2}\frac{\partial}{\partial x_m}\psi(x_1, \ldots, x_N). \tag{5.4}$$

Both representatives are well defined on the *enlarged* space $\mathcal{T}(\mathcal{H})$ and, for the sake of simplicity, we have used the same symbol for the algebra element x_m and the independent variable of the mth $\mathcal{H}$ space. Since in the following we will never write explicitly abstract elements of the position–momentum Heisenberg algebra, this will cause no confusion. In general, in order to have simple notation, from now on we will indicate with the same symbol all the algebra elements and all their representative operators, as the distinction will arise from the context. This implies an accidental coincidence of the symbols for the independent position variable and the corresponding position representative operator, but we will never write explicitly position representative operators in the following: x_m will always indicate exclusively the position variable.

From (5.2) and (5.3), (5.4) we have the following representation of $V_m^{\pm}$ ($1 \leqslant m \leqslant N$):

$$(V_m^{\pm}\psi)(x_1, \ldots, x_N) = \left[\pm(x_{m+1} - x_m) + \frac{\mathrm{i}\pi\beta^2}{2}\frac{\partial}{\partial x_m}\right]\psi(x_1, \ldots, x_N) \tag{5.5}$$

where the independent variable inherits the identification $x_{N+1} = x_1$ from the algebra element. This implies that the operator representatives of $W_m^{\pm} = \mathrm{e}^{\mathrm{i}V_m^{\pm}}$ ($1 \leqslant m \leqslant N$) are defined as unitary operators acting on $\mathcal{T}(\mathcal{H})$ as follows:

$$[W_m^{\pm}\psi](x_1, \ldots, x_N) = \mathrm{e}^{\pm\mathrm{i}(x_{m+1}-x_m)}\,\mathrm{e}^{\pm\frac{\mathrm{i}\pi\beta^2}{4}}\,\psi\left(x_1, \ldots, x_m - \frac{\pi\beta^2}{2}, \ldots, x_N\right) \tag{5.6}$$

with the usual prescription $x_{N+1} = x_1$, for $m = N$.

Finally, inserting (5.6) in (3.19) we obtain a coordinate representation for the left and right Lax operators. Since the entries of the Lax operators depend on $W_m^{\pm}$, they are well-defined unitary operators acting on the whole $\mathcal{T}(\mathcal{H})$.

Let us finally remark that the definition of the representation is a crucial problem in usual ABA and *a fortiori* in our non-ultralocal case: actually, this is the origin of the mistake in [35].

6. Algebraic Bethe ansatz in the conformal case

6.1. The left monodromy matrix

In this subsection we will consider the left conformal monodromy matrix (4.22) whose entries are defined by

$$M(\lambda) = L_N(\lambda) \dots L_1(\lambda) \equiv \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \tag{6.1}$$

We will consider the case of an even number of sites, that is $N \in 2\mathbb{N}$, and write the Bethe equations, eigenvalues and eigenvectors of its transfer matrix by developing an extension of ABA techniques. In fact, the usual ABA grounds on the usual Yang–Baxter equation and hence we have to modify it in such a way that we can use the braided equation efficiently. This can be rigorously done using the coordinate representation given in the previous section.

Let us define as in [35] the fused Lax operator and its entries: $k \in 2\mathbb{N}$, $2 \leqslant k \leqslant N$,

$$F_k(\lambda) \equiv L_k(\lambda) L_{k-1}(\lambda) \equiv \begin{pmatrix} F_{11}^{(k,k-1)}(\lambda) & F_{12}^{(k,k-1)}(\lambda) \\ F_{21}^{(k,k-1)}(\lambda) & F_{22}^{(k,k-1)}(\lambda) \end{pmatrix}. \tag{6.2}$$

Hence, from definition (3.19), the entries are given by

$$F_{11}^{(k,k-1)}(\lambda) = (W_k^-)^{-1}(W_{k-1}^-)^{-1} + \Delta^2\lambda^2 W_k^+ (W_{k-1}^+)^{-1} \tag{6.3}$$

$$F_{12}^{(k,k-1)}(\lambda) = \Delta\lambda \left[(W_k^-)^{-1} W_{k-1}^+ + W_k^+ W_{k-1}^-\right] \tag{6.4}$$

$$F_{21}^{(k,k-1)}(\lambda) = \Delta\lambda \left[(W_k^+)^{-1}(W_{k-1}^-)^{-1} + W_k^- (W_{k-1}^+)^{-1}\right] \tag{6.5}$$

$$F_{22}^{(k,k-1)}(\lambda) = W_k^- W_{k-1}^- + \Delta^2\lambda^2 (W_k^+)^{-1} W_{k-1}^+. \tag{6.6}$$

Let us now consider the coordinate representation (5.6). The fused Lax operator entries (6.3), (6.5), (6.6) act on the representation space $\mathcal{T}(\mathcal{H})$ as follows:

$$\begin{aligned}\left[F_{11}^{(k,k-1)}(\lambda)\psi\right](x_1,\dots,x_N) &= \mathrm{e}^{\mathrm{i}(x_{k+1}-x_{k-1})}\psi(x_1,\dots,x_{k-1}^+,x_k^+,\dots,x_N)\\ &\quad + \Delta^2\lambda^2 q^{-1}\,\mathrm{e}^{\mathrm{i}(x_{k+1}+x_{k-1}-2x_k)}\psi(x_1,\dots,x_{k-1}^+,x_k^-,\dots,x_N)\end{aligned} \tag{6.7}$$

$$\begin{aligned}\left[F_{22}^{(k,k-1)}(\lambda)\psi\right](x_1,\dots,x_N) &= \mathrm{e}^{-\mathrm{i}(x_{k+1}-x_{k-1})}\psi(x_1,\dots,x_{k-1}^-,x_k^-,\dots,x_N)\\ &\quad + \Delta^2\lambda^2 q^{-1}\,\mathrm{e}^{-\mathrm{i}(x_{k+1}+x_{k-1}-2x_k)}\psi(x_1,\dots,x_{k-1}^-,x_k^+,\dots,x_N)\end{aligned} \tag{6.8}$$

$$\begin{aligned}\left[F_{21}^{(k,k-1)}(\lambda)\psi\right](x_1,\dots,x_N) &= \Delta\lambda q^{-\frac{1}{2}}\left[\mathrm{e}^{-\mathrm{i}(x_{k+1}-x_{k-1})}\psi(x_1,\dots,x_{k-1}^+,x_k^-,\dots,x_N)\right.\\ &\quad \left.+\,\mathrm{e}^{-\mathrm{i}(x_{k+1}+x_{k-1}-2x_k)}\psi(x_1,\dots,x_{k-1}^+,x_k^+,\dots,x_N)\right]\end{aligned} \tag{6.9}$$

where for the sake of conciseness we have defined

$$x_k^{\pm} \equiv x_k \pm \pi\beta^2/2 \tag{6.10}$$

and, of course, the variable x_{N+1} is identified with x_1. Note from the previous formulae that the action of the operator $F_{ij}^{(k,k-1)}$ is not confined to the coordinate (x_k, x_{k-1}) and is therefore called *non-ultralocal*.

In order to carry on the usual ABA procedure, we have to find the so-called *pseudovacuum* states.

Definition 2. *In a fixed representation a* pseudovacuum *or* false vacuum *is a vector which is a simultaneous eigenstate of the diagonal elements $A(\lambda)$ and $D(\lambda)$ of the monodromy matrix and which is annihilated by the off-diagonal element $C(\lambda)$, for every $\lambda \in \mathbb{C}$.*

We are now in a position to show that in the coordinate representation space $\mathcal{T}(\mathcal{H})$ the pseudovacua are given by

$$\Omega(x_1, \ldots, x_N) = \prod_{\substack{k=2 \\ k\in 2\mathbb{Z}}}^{N} f(x_{k-1} - x_k) \tag{6.11}$$

where f is an element of $\mathcal{H}\otimes\mathcal{H}$, depending on the difference of the coordinates and satisfying the shift property:

$$f(x + \pi\beta^2) = -\mathrm{e}^{-2\mathrm{i}x} f(x). \tag{6.12}$$

The functional equation (6.12) possesses in general infinite solutions, for instance

$$f(x) = \exp\left(-\frac{\mathrm{i}x^2}{\pi\beta^2} + \mathrm{i}x + \frac{\mathrm{i}x}{\beta^2}\right) \tag{6.13}$$

and functions obtained from it by multiplication by a periodic function with period $\pi\beta^2$. As we will show, however, every solution of (6.12) gives a pseudovacuum with the same eigenvalue for A and D. Hence, we do not need to single out any specific solution of (6.12).

The proof of the fact that (6.11) with (6.12) is a pseudovacuum, relies on annihilation properties following immediately from (6.9) and (6.12):

$$\left[F_{21}^{(k,k-1)}(\lambda)\Omega\right](x_1, \ldots, x_N) = 0. \tag{6.14}$$

Indeed, let us consider the expressions of A, D, C in terms of the elements of the fused Lax operator. For example, if $N = 6$ we have (understanding the dependence on the spectral parameter):

$$\begin{aligned}
A &= F_{11}^{(6,5)}\left[F_{11}^{(4,3)}F_{11}^{(2,1)} + F_{12}^{(4,3)}F_{21}^{(2,1)}\right] + F_{12}^{(6,5)}\left[F_{21}^{(4,3)}F_{11}^{(2,1)} + F_{22}^{(4,3)}F_{21}^{(2,1)}\right] \\
D &= F_{21}^{(6,5)}\left[F_{11}^{(4,3)}F_{12}^{(2,1)} + F_{12}^{(4,3)}F_{22}^{(2,1)}\right] + F_{22}^{(6,5)}\left[F_{21}^{(4,3)}F_{12}^{(2,1)} + F_{22}^{(4,3)}F_{22}^{(2,1)}\right] \\
C &= F_{21}^{(6,5)}\left[F_{11}^{(4,3)}F_{11}^{(2,1)} + F_{12}^{(4,3)}F_{21}^{(2,1)}\right] + F_{22}^{(6,5)}\left[F_{21}^{(4,3)}F_{11}^{(2,1)} + F_{22}^{(4,3)}F_{21}^{(2,1)}\right].
\end{aligned} \tag{6.15}$$

Now, we prove, by using the W's exchange algebra (3.18), some very fundamental exchange relations between the $F_{ij}^{(k,k-1)}(\lambda)$ ($k \in 2\mathbb{N}$, $2 \leqslant k \leqslant N$)—not necessarily in a representation:

- exchange (21)–(11)

$$\begin{aligned}
F_{21}^{(k+2,k+1)}(\lambda)F_{11}^{(k,k-1)}(\lambda') &= q^{-\frac{1}{2}}F_{11}^{(k,k-1)}(\lambda')F_{21}^{(k+2,k+1)}(\lambda) \\
F_{21}^{(N,N-1)}(\lambda)F_{11}^{(2,1)}(\lambda') &= q^{-\frac{1}{2}}F_{11}^{(2,1)}(\lambda')F_{21}^{(N,N-1)}(\lambda)
\end{aligned} \tag{6.16}$$

- exchange (21)–(12)

$$
\begin{aligned}
F_{21}^{(k+2,k+1)}(\lambda)F_{12}^{(k,k-1)}(\lambda') &= q^{-\frac{1}{2}}F_{12}^{(k,k-1)}(\lambda')F_{21}^{(k+2,k+1)}(\lambda) \\
F_{21}^{(N,N-1)}(\lambda)F_{12}^{(2,1)}(\lambda') &= q^{\frac{1}{2}}F_{12}^{(2,1)}(\lambda')F_{21}^{(N,N-1)}(\lambda)
\end{aligned}
\tag{6.17}
$$

- exchange (21)–(22)

$$
\begin{aligned}
F_{21}^{(k+2,k+1)}(\lambda)F_{22}^{(k,k-1)}(\lambda') &= q^{\frac{1}{2}}F_{22}^{(k,k-1)}(\lambda')F_{21}^{(k+2,k+1)}(\lambda) \\
F_{21}^{(N,N-1)}(\lambda)F_{22}^{(2,1)}(\lambda') &= q^{\frac{1}{2}}F_{22}^{(2,1)}(\lambda')F_{21}^{(N,N-1)}(\lambda)
\end{aligned}
\tag{6.18}
$$

- commutation if ($k' \in 2\mathbb{N}, 2 \leqslant k' \leqslant N$) $2 < |k-k'| < N-2$

$$
\left[F_{ij}^{(k,k-1)}(\lambda), F_{i'j'}^{(k',k'-1)}(\lambda')\right] = 0. \tag{6.19}
$$

Consequently, through the exchange properties (6.16)–(6.19), we can bring all the factors $F_{21}^{(k,k-1)}$ to the right of the addenda in the expressions of $A(\lambda)$, $D(\lambda)$, $C(\lambda)$. The following action of $A(\lambda)$, $D(\lambda)$, $C(\lambda)$ on the state Ω (6.11) is a consequence of their form (see, e.g., formulae (6.15) in the case $N = 6$) and of annihilation properties (6.14):

$$
A(\lambda)\Omega = \prod_{\substack{k=2\\k\in 2\mathbb{Z}}}^{\overset{N}{\leftarrow}} F_{11}^{(k,k-1)}(\lambda)\Omega \qquad D(\lambda)\Omega = \prod_{\substack{k=2\\k\in 2\mathbb{Z}}}^{\overset{N}{\leftarrow}} F_{22}^{(k,k-1)}(\lambda)\Omega \qquad C(\lambda)\Omega = 0 \tag{6.20}
$$

where the arrow $\leftarrow$ indicates the direction of increasing indices in the ordered product. We are left with proving that Ω is a simultaneous eigenvector of $A(\lambda)$ and $D(\lambda)$: this will be realized by the following theorem and its corollary.

Theorem 2. *The action of the ordered product of diagonal elements of the fused Lax operators (6.2) on the states (6.11) is the following* ($k \leqslant N$)*:*

$$
\left[\prod_{\substack{h=2\\h\in 2\mathbb{Z}}}^{\overset{k}{\leftarrow}} F_{11}^{(h,h-1)}(\lambda)\Omega\right](x_1,\ldots,x_N) = \mathrm{e}^{\mathrm{i}(x_{k+1}-x_1)}q^{-\frac{1}{2}(\frac{k}{2}-1)}(1-\Delta^2\lambda^2q^{-1})^{\frac{k}{2}}\Omega(x_1,\ldots,x_N)
$$

$$
\left[\prod_{\substack{h=2\\h\in 2\mathbb{Z}}}^{\overset{k}{\leftarrow}} F_{22}^{(h,h-1)}(\lambda)\Omega\right](x_1,\ldots,x_N) = \mathrm{e}^{-\mathrm{i}(x_{k+1}-x_1)}q^{-\frac{1}{2}(\frac{k}{2}-1)}(1-\Delta^2\lambda^2q)^{\frac{k}{2}}\Omega(x_1,\ldots,x_N).
$$

Proof. Let us show by induction the first formula. For $k = 2$ it follows from (6.7) and the shift property (6.12) . For general $k \leqslant N$ we have from (6.7)

$$
\begin{aligned}
&\left[\prod_{\substack{h=2\\h\in 2\mathbb{Z}}}^{\overset{k}{\leftarrow}} F_{11}^{(h,h-1)}(\lambda)\Omega\right](x_1,\ldots,x_N) \\
&\quad = \mathrm{e}^{\mathrm{i}(x_{k+1}-x_{k-1})}\left[\prod_{\substack{h=2\\h\in 2\mathbb{Z}}}^{\overset{k-2}{\leftarrow}} F_{11}^{(h,h-1)}(\lambda)\Omega\right](x_1,\ldots,x_{k-2},x_{k-1}^+,x_k^+,x_{k+1},\ldots,x_N) \\
&\quad + \Delta^2\lambda^2q^{-1}\,\mathrm{e}^{\mathrm{i}(x_{k+1}+x_{k-1}-2x_k)}\left[\prod_{\substack{h=2\\h\in 2\mathbb{Z}}}^{\overset{k-2}{\leftarrow}} F_{11}^{(h,h-1)}(\lambda)\Omega\right] \\
&\quad \times (x_1,\ldots,x_{k-2},x_{k-1}^+,x_k^-,x_{k+1},\ldots,x_N).
\end{aligned}
$$

Applying the inductive hypothesis we get

$$\left[\overset{k}{\underset{\substack{h=2\\h\in 2\mathbb{Z}}}{\overleftarrow{\prod}}} F_{11}^{(h,h-1)}(\lambda)\Omega\right](x_1,\ldots,x_N)$$
$$= \mathrm{e}^{\mathrm{i}(x_{k+1}-x_{k-1})}\,\mathrm{e}^{\mathrm{i}(x_{k-1}^{+}-x_1)}q^{-\frac{1}{2}\left(\frac{k}{2}-2\right)}(1-\Delta^2\lambda^2q^{-1})^{\frac{k}{2}-1}\Omega(x_1,\ldots,x_N)$$
$$+\Delta^2\lambda^2q^{-1}\,\mathrm{e}^{\mathrm{i}(x_{k+1}+x_{k-1}-2x_k)}\,\mathrm{e}^{\mathrm{i}(x_{k-1}^{+}-x_1)}q^{-\frac{1}{2}\left(\frac{k}{2}-2\right)}(1-\Delta^2\lambda^2q^{-1})^{\frac{k}{2}-1}$$
$$\times\,\Omega(x_1,\ldots,x_{k-2},x_{k-1}^{+},x_k^{-},x_{k+1},\ldots,x_N). \tag{6.21}$$

Using the shift property (6.12) in the last term gives

$$\Omega(x_1,\ldots,x_{k-2},x_{k-1}^{+},x_k^{-},x_{k+1},\ldots,x_N) = -\mathrm{e}^{2\mathrm{i}(x_k-x_{k-1})}\Omega(x_1,\ldots,x_N). \tag{6.22}$$

Hence the two terms on the right-hand side are proportional. After gathering them, we get the first formula of theorem 2.

The second formula follows in an analogous way, after using the shift property (6.12) in the form

$$f(x-\pi\beta^2) = -\mathrm{e}^{2\mathrm{i}(x-\pi\beta^2)}f(x). \tag{6.23}$$

□

Corollary 2. *The states (6.11) are eigenvectors of the elements $A(\lambda)$ and $D(\lambda)$ of the left conformal monodromy matrix (4.22). The corresponding common eigenvalues are given by the formulae*

$$[A(\lambda)\Omega] = q^{-\frac{1}{2}\left(\frac{N}{2}-1\right)}(1-\Delta^2\lambda^2q^{-1})^{\frac{N}{2}}\Omega \equiv \rho_N(\lambda)\Omega \tag{6.24}$$
$$[D(\lambda)\Omega] = q^{-\frac{1}{2}\left(\frac{N}{2}-1\right)}(1-\Delta^2\lambda^2q)^{\frac{N}{2}}\Omega \equiv \sigma_N(\lambda)\Omega. \tag{6.25}$$

Proof. The proof follows from theorem 2 for $k=N$, recalling that $x_{N+1}=x_1$. □

Eventually, formulae (6.20), (6.24) and (6.25) show that the states (6.11) are pseudovacua of the monodromy matrix (4.22) with the same $A(\lambda)$ and $D(\lambda)$ eigenvalues for any $f(x)$ verifying (6.12). Nevertheless, we need to note that the two-site state $f(x_{k-1}-x_k)$ is not a pseudovacuum for $A^{(k,k-1)}\equiv F_{11}^{(k,k-1)}$, $D^{(k,k-1)}\equiv F_{22}^{(k,k-1)}$, $C^{(k,k-1)}\equiv F_{21}^{(k,k-1)}$: this property is quite rare and called the non-ultralocality of the pseudovacuum.

Let us now derive the Bethe ansatz equations. From (4.4) it follows that the left conformal monodromy matrix (4.22) satisfies the braided Yang–Baxter relation

$$R_{ab}\left(\frac{\lambda}{\lambda'}\right)M_a(\lambda)Z_{ab}^{-1}M_b(\lambda') = M_b(\lambda')Z_{ab}^{-1}M_a(\lambda)R_{ab}\left(\frac{\lambda}{\lambda'}\right) \tag{6.26}$$

which contains implicitly these exchange rules between $B(\lambda')$ and $A(\lambda)$, $D(\lambda)$, respectively:

$$A(\lambda)B(\lambda') = \frac{q^{-1}}{a\left(\frac{\lambda'}{\lambda}\right)}B(\lambda')A(\lambda) - q^{-1}\frac{b\left(\frac{\lambda'}{\lambda}\right)}{a\left(\frac{\lambda'}{\lambda}\right)}B(\lambda)A(\lambda') \tag{6.27}$$
$$D(\lambda)B(\lambda') = \frac{q}{a\left(\frac{\lambda}{\lambda'}\right)}B(\lambda')D(\lambda) - q\frac{b\left(\frac{\lambda}{\lambda'}\right)}{a\left(\frac{\lambda}{\lambda'}\right)}B(\lambda)D(\lambda'). \tag{6.28}$$

In equations (6.27), (6.28) we have defined for the sake of conciseness

$$a(\xi) = \frac{\xi^{-1}-\xi}{q^{-1}\xi^{-1}-q\xi} \qquad b(\xi) = \frac{q^{-1}-q}{q^{-1}\xi^{-1}-q\xi}. \tag{6.29}$$

Note the presence of the factors $q^{\pm 1}$ in expressions (6.27), (6.28): they come from the matrix Z_{ab} and represent the contribution to exchange relations coming from non-ultralocality. Now, as usual, we build Bethe states

$$\Psi(\lambda_1, \ldots, \lambda_l) = \prod_{r=1}^{l} B(\lambda_r)\Omega \tag{6.30}$$

acting on a pseudovacuum with the *creators of pseudoparticles* $B(\lambda_r)$, without considering the ordering, because of the commuting property encoded in the braided Yang–Baxter equation:

$$[B(\lambda), B(\lambda')] = 0. \tag{6.31}$$

From (6.27), (6.28) we find the action of $A(\lambda)$ and $D(\lambda)$ on Bethe states:

$$A(\lambda)\Psi(\lambda_1, \ldots, \lambda_l) = q^{-l} \prod_{r=1}^{l} \frac{1}{a\left(\frac{\lambda_r}{\lambda}\right)} \rho_N(\lambda)\Psi(\lambda_1, \ldots, \lambda_l) + \cdots \tag{6.32}$$

$$D(\lambda)\Psi(\lambda_1, \ldots, \lambda_l) = q^{l} \prod_{r=1}^{l} \frac{1}{a\left(\frac{\lambda}{\lambda_r}\right)} \sigma_N(\lambda)\Psi(\lambda_1, \ldots, \lambda_l) + \cdots. \tag{6.33}$$

The dots in (6.32), (6.33) indicate extra terms which are not proportional to state (6.30). Hence, in general, states (6.30) are not eigenstates of the λ-dependent transfer matrices $T(\lambda) = A(\lambda) + D(\lambda)$. This is true if and only if the set of complex numbers $\{\lambda_1, \ldots, \lambda_l\}$ satisfies the following Bethe equations (BEs):

$$q^{-l} \prod_{\substack{r=1 \\ r\neq s}}^{l} \frac{1}{a\left(\frac{\lambda_r}{\lambda_s}\right)} \rho_N(\lambda_s) = q^{l} \prod_{\substack{r=1 \\ r\neq s}}^{l} \frac{1}{a\left(\frac{\lambda_s}{\lambda_r}\right)} \sigma_N(\lambda_s). \tag{6.34}$$

By using the expressions for $\rho_N(\lambda)$ and $\sigma_N(\lambda)$ coming from (6.24), (6.25) and for $a(\lambda)$ coming from (6.29), we can rewrite the BEs as follows:

$$q^{-2l} \prod_{\substack{r=1 \\ r\neq s}}^{l} \frac{q\lambda_r^2 - q^{-1}\lambda_s^2}{q^{-1}\lambda_r^2 - q\lambda_s^2} = \left(\frac{1 - \Delta^2\lambda_s^2 q}{1 - \Delta^2\lambda_s^2 q^{-1}}\right)^{N/2}. \tag{6.35}$$

The definition

$$\Delta\lambda_r \equiv \mathrm{e}^{\alpha_r} \tag{6.36}$$

allows us to rewrite BEs (6.35) in the more diffuse trigonometric form:

$$\prod_{\substack{r=1 \\ r\neq s}}^{l} \frac{\sinh\left(\alpha_s - \alpha_r + \mathrm{i}\pi\beta^2\right)}{\sinh\left(\alpha_s - \alpha_r - \mathrm{i}\pi\beta^2\right)} = \left[\frac{\sinh\left(\alpha_s - \frac{\mathrm{i}\pi\beta^2}{2}\right)}{\sinh\left(\alpha_s + \frac{\mathrm{i}\pi\beta^2}{2}\right)}\right]^{N/2} \mathrm{e}^{-\frac{\mathrm{i}\pi\beta^2}{2}N - 2\mathrm{i}\pi\beta^2 l}. \tag{6.37}$$

Eventually, let us deduce, from equations (6.32), (6.33), the eigenvalues of the left transfer matrix $T(\lambda) \equiv \mathrm{Tr}\, M(\lambda)$, relative to Bethe states (6.30), (6.35):

$$\Lambda(\lambda, \{\lambda_r\}) = q^{-l} \prod_{r=1}^{l} \frac{1}{a\left(\frac{\lambda_r}{\lambda}\right)} \rho_N(\lambda) + q^{l} \prod_{r=1}^{l} \frac{1}{a\left(\frac{\lambda}{\lambda_r}\right)} \sigma_N(\lambda). \tag{6.38}$$

By using the expressions for $\rho_N(\lambda)$ and $\sigma_N(\lambda)$ coming from (6.24), (6.25), we write (6.38) in the following way:

$$\Lambda(\lambda,\{\lambda_r\}) = q^{-l}\prod_{r=1}^{l}\frac{q^{-1}\lambda^2-q\lambda_r^2}{\lambda^2-\lambda_r^2}q^{-\frac{1}{2}(\frac{N}{2}-1)}(1-\Delta^2\lambda^2q^{-1})^{N/2} + q^{l}\prod_{r=1}^{l}\frac{q\lambda^2-q^{-1}\lambda_r^2}{\lambda^2-\lambda_r^2}q^{-\frac{1}{2}(\frac{N}{2}-1)}(1-\Delta^2\lambda^2q)^{N/2}. \tag{6.39}$$

Let us make some comments on the results of this subsection. The BEs (6.37) are the equations for a spin chain of spin $-\frac{1}{2}$ with, in addition, the *twist* $\mathrm{e}^{-\frac{\mathrm{i}\pi\beta^2}{2}N-2\mathrm{i}\pi\beta^2 l}$. Instead, in paper [35] they turn out to be of different signs (spin $+\frac{1}{2}$ chain), because of an inconsistent definition of the pseudovacuum, affecting also the final expressions of the eigenvalues. As far as we know, the presence of the l-dependent twist appearing in the BEs is a new feature and is a direct consequence of non-ultralocality, encoded in the Z_{ab} matrix. In view of the fact that this twist depends on the number of Bethe roots (solutions of the BEs), it will be said to be *dynamically generated*. The forms of the eigenvalues of the transfer matrix (6.39) are also those of a dynamically twisted $-\frac{1}{2}$ spin chain. A similarly generated twist appeared in [38] in the case of a CFT—Liouville theory—but it only depends on the number of sites N. Besides, in paper [38] a detailed analysis has been carried out to conjecture a one-to-one correspondence between Bethe states and squares in the Kac table of minimal CFTs. These facts lead us to think that the (cylinder) continuum limit of equations (6.37), (6.39) describes the chiral sector of CFTs and their chiral IMI encoded in the transfer matrix. In a forthcoming paper [29] we will examine the (cylinder) continuum limit for special values of β^2 corresponding to the very interesting case of minimal CFTs in order to prove this conjecture.

6.2. Right monodromy matrix

We could repeat all the steps and considerations of the last subsection in the case of the right conformal monodromy matrix (4.23),

$$\bar{M}(\lambda) = \bar{L}_N(\lambda^{-1})\ldots\bar{L}_1(\lambda^{-1}) \equiv \begin{pmatrix}\bar{A}(\lambda) & \bar{B}(\lambda)\\ \bar{C}(\lambda) & \bar{D}(\lambda)\end{pmatrix} \qquad N\in 2\mathbb{N} \tag{6.40}$$

but we will briefly illustrate them, since the main conclusions can be obtained from the results of the last section using the duality connecting left and right Lax operators.

Indeed, the fused Lax operator and its entries are defined by ($k\in 2\mathbb{N}, 2\leqslant k\leqslant N$)

$$\bar{F}_k(\lambda^{-1}) \equiv \bar{L}_k(\lambda^{-1})\bar{L}_{k-1}(\lambda^{-1}) \equiv \begin{pmatrix}\bar{F}_{11}^{(k,k-1)}(\lambda^{-1}) & \bar{F}_{12}^{(k,k-1)}(\lambda^{-1})\\ \bar{F}_{21}^{(k,k-1)}(\lambda^{-1}) & \bar{F}_{22}^{(k,k-1)}(\lambda^{-1})\end{pmatrix} \tag{6.41}$$

and hence, using the duality connecting left and right Lax operators (3.19), the entries of (6.41) are given by the entries of (6.2) in which W_k^+ is exchanged with W_k^- and λ is exchanged with λ^{-1}.

However, for clarity and future applications we write in the coordinate representation (5.6) the action of the following fused Lax operator entries on the space $\mathcal{T}(\mathcal{H})$:

$$\left[\bar{F}_{11}^{(k,k-1)}(\lambda^{-1})\psi\right](x_1,\ldots,x_N) = \mathrm{e}^{-\mathrm{i}(x_{k+1}-x_{k-1})}\psi(x_1,\ldots,x_{k-1}^+,x_k^+,\ldots,x_N) + \Delta^2\lambda^{-2}q\,\mathrm{e}^{-\mathrm{i}(x_{k+1}+x_{k-1}-2x_k)}\psi(x_1,\ldots,x_{k-1}^+,x_k^-,\ldots,x_N) \tag{6.42}$$

$$\left[\bar{F}_{22}^{(k,k-1)}(\lambda^{-1})\psi\right](x_1,\ldots,x_N) = \mathrm{e}^{\mathrm{i}(x_{k+1}-x_{k-1})}\psi(x_1,\ldots,x_{k-1}^-,x_k^-,\ldots,x_N) + \Delta^2\lambda^{-2}q\,\mathrm{e}^{\mathrm{i}(x_{k+1}+x_{k-1}-2x_k)}\psi(x_1,\ldots,x_{k-1}^-,x_k^+,\ldots,x_N) \tag{6.43}$$

$$\left[\bar{F}_{21}^{(k,k-1)}(\lambda^{-1})\psi\right](x_1,\ldots,x_N) = \Delta\lambda^{-1}q^{\frac{1}{2}}\left[\mathrm{e}^{\mathrm{i}(x_{k+1}-x_{k-1})}\psi(x_1,\ldots,x_{k-1}^+,x_k^-,\ldots,x_N) + \mathrm{e}^{\mathrm{i}(x_{k+1}+x_{k-1}-2x_k)}\psi(x_1,\ldots,x_{k-1}^+,x_k^+,\ldots,x_N)\right] \tag{6.44}$$

where again x_{N+1} is to be meant as x_1.

We want to remark also that using these formulae one can find in the coordinate representation space the explicit form of the pseudovacua:

$$\bar{\Omega}(x_1,\ldots,x_N) = \prod_{\substack{k=2\\k\in 2\mathbb{Z}}}^{N} f(x_{k-1}-x_k)^{-1} \tag{6.45}$$

where $f(x)$ is a non-zero solution of (6.12).

The use of the aforementioned duality allows us to write easily the Bethe equations and the eigenvalues of the right conformal transfer matrix on the Bethe states:

$$\bar{\Psi}(\lambda_1,\ldots,\lambda_l) = \prod_{r=1}^{l}\bar{B}(\lambda_r)\bar{\Omega}. \tag{6.46}$$

Indeed, they can be obtained from the corresponding left formulae (6.35), (6.39) by simply replacing q with q^{-1} and λ, λ_r with $\lambda^{-1}, \lambda_r^{-1}$. Explicitly, we obtain that the Bethe equations for the right conformal monodromy matrix read

$$q^{2l}\prod_{\substack{r=1\\r\neq s}}^{l}\frac{q\lambda_r^2-q^{-1}\lambda_s^2}{q^{-1}\lambda_r^2-q\lambda_s^2} = \left(\frac{1-\Delta^2\lambda_s^{-2}q^{-1}}{1-\Delta^2\lambda_s^{-2}q}\right)^{N/2} \tag{6.47}$$

or in a trigonometric form $\left(\Delta^{-1}\lambda_r \equiv \mathrm{e}^{\bar{\alpha}_r}\right)$:

$$\prod_{\substack{r=1\\r\neq s}}^{l}\frac{\sinh\left(\bar{\alpha}_s-\bar{\alpha}_r+\mathrm{i}\pi\beta^2\right)}{\sinh\left(\bar{\alpha}_s-\bar{\alpha}_r-\mathrm{i}\pi\beta^2\right)} = \left[\frac{\sinh\left(\bar{\alpha}_s-\frac{\mathrm{i}\pi\beta^2}{2}\right)}{\sinh\left(\bar{\alpha}_s+\frac{\mathrm{i}\pi\beta^2}{2}\right)}\right]^{N/2}\mathrm{e}^{\frac{\mathrm{i}\pi\beta^2}{2}N+2\mathrm{i}\pi\beta^2 l}. \tag{6.48}$$

In addition, the eigenvalues of the transfer matrix $\bar{T}(\lambda) \equiv \mathrm{Tr}\,\bar{M}(\lambda)$ are

$$\bar{\Lambda}(\lambda,\{\lambda_r\}) = q^{l}\prod_{r=1}^{l}\frac{q^{-1}\lambda^2-q\lambda_r^2}{\lambda^2-\lambda_r^2}q^{\frac{1}{2}(\frac{N}{2}-1)}(1-\Delta^2\lambda^{-2}q)^{N/2} + q^{-l}\prod_{r=1}^{l}\frac{q\lambda^2-q^{-1}\lambda_r^2}{\lambda^2-\lambda_r^2}q^{\frac{1}{2}(\frac{N}{2}-1)}(1-\Delta^2\lambda^{-2}q^{-1})^{N/2}. \tag{6.49}$$

We can comment on the results of this subsection in an analogous way as we have done at the end of the previous subsection, after taking into account the change of left (chiral) to right (anti-chiral).

7. Algebraic Bethe ansatz in the off-critical case

The minimal CFTs perturbed by the primary field $\Phi_{1,3}$ possess local IMI, which are suitable *deformations* of those in left and right quantum KdV theory in the continuum limit

[17, 20, 27, 28]. For this reason, we *couple together* the left and right theories with the aim of describing a lattice discretization (or better regularization) of perturbed CFTs. Preserving integrability, we would like to conjecture that this coupling of the left (chiral) and right (anti-chiral) sectors is realized equivalently by the monodromy matrices (4.24) or (4.25), which contain both L_m and $\bar{L}_m$ and verify the braided Yang–Baxter relation. In this section we will diagonalize the associated transfer matrices by means of an extended version of algebraic Bethe ansatz techniques.

Let us start with the monodromy matrix (4.24) and define its entries ($N \in 4\mathbb{N}$)

$$\mathbf{M}(\lambda) = \overleftarrow{\prod_{i=1}^{N/4}} \bar{F}_{4i}\left(\frac{\mu^{\frac{1}{2}}}{\lambda}\right) F_{4i-2}\left(\lambda\mu^{\frac{1}{2}}\right) \equiv \begin{pmatrix} \mathbf{A}(\lambda;\mu) & \mathbf{B}(\lambda;\mu) \\ \mathbf{C}(\lambda;\mu) & \mathbf{D}(\lambda;\mu) \end{pmatrix}. \tag{7.1}$$

We want to write the eigenvectors and eigenvalues of the transfer matrix in terms of the solutions (roots) of the Bethe equations (BEs).

Remember that the fused Lax operators in (7.1) are

$$\bar{F}_{4i}(\lambda) = \bar{L}_{4i}(\lambda)\bar{L}_{4i-1}(\lambda) \qquad F_{4i-2}(\lambda) = L_{4i-2}(\lambda)L_{4i-3}(\lambda) \tag{7.2}$$

and that their entries, defined by (6.41) for $\bar{F}_k$ and (6.2) for F_k, are explicitly given by

$$\bar{F}_{11}^{(4i,4i-1)}(\lambda) = (W_{4i}^+)^{-1}(W_{4i-1}^+)^{-1} + \Delta^2\lambda^2 W_{4i}^-(W_{4i-1}^-)^{-1} \tag{7.3}$$

$$\bar{F}_{12}^{(4i,4i-1)}(\lambda) = \Delta\lambda\left[(W_{4i}^+)^{-1}W_{4i-1}^- + W_{4i}^- W_{4i-1}^+\right] \tag{7.4}$$

$$\bar{F}_{21}^{(4i,4i-1)}(\lambda) = \Delta\lambda\left[(W_{4i}^-)^{-1}(W_{4i-1}^+)^{-1} + W_{4i}^+(W_{4i-1}^-)^{-1}\right] \tag{7.5}$$

$$\bar{F}_{22}^{(4i,4i-1)}(\lambda) = W_{4i}^+ W_{4i-1}^+ + \Delta^2\lambda^2(W_{4i}^-)^{-1}W_{4i-1}^- \tag{7.6}$$

$$F_{11}^{(4i-2,4i-3)}(\lambda) = (W_{4i-2}^-)^{-1}(W_{4i-3}^-)^{-1} + \Delta^2\lambda^2 W_{4i-2}^+(W_{4i-3}^+)^{-1} \tag{7.7}$$

$$F_{12}^{(4i-2,4i-3)}(\lambda) = \Delta\lambda\left[(W_{4i-2}^-)^{-1}W_{4i-3}^+ + W_{4i-2}^+ W_{4i-3}^-\right] \tag{7.8}$$

$$F_{21}^{(4i-2,4i-3)}(\lambda) = \Delta\lambda\left[(W_{4i-2}^+)^{-1}(W_{4i-3}^-)^{-1} + W_{4i-2}^-(W_{4i-3}^+)^{-1}\right] \tag{7.9}$$

$$F_{22}^{(4i-2,4i-3)}(\lambda) = W_{4i-2}^- W_{4i-3}^- + \Delta^2\lambda^2(W_{4i-2}^+)^{-1}W_{4i-3}^+. \tag{7.10}$$

We now consider the coordinate representation. Actually, we have already written how the operator representatives of (7.3), (7.5), (7.6) and (7.7), (7.9), (7.10) act on the coordinate space $\mathcal{T}(\mathcal{H})$ in formulae (6.7)–(6.9) and (6.42)–(6.44). These are the entries which are important for our calculations.

What are now different are the pseudovacua. Indeed, we want to show that in the coordinate representation the pseudovacua are given by the following element of $\mathcal{T}(\mathcal{H})$:

$$\mathbf{\Omega}(x_1,\ldots,x_N) = \prod_{i=1}^{N/4} f(x_{4i-1}-x_{4i})^{-1} f(x_{4i-3}-x_{4i-2})\delta\left(\sum_{i=1}^{N/4}(x_{4i-3}-x_{4i-1})\right) \tag{7.11}$$

where the function $f(x)$ is a solution of (6.12).

Let us prove this statement in some steps. These annihilation properties, derived from (6.9), (6.12), (6.44), are crucial:

$$\left[\bar{F}_{21}^{(4i,4i-1)}(\lambda)\mathbf{\Omega}\right](x_1,\ldots,x_N) = 0 \qquad \left[F_{21}^{(4i-2,4i-3)}(\lambda)\mathbf{\Omega}\right](x_1,\ldots,x_N) = 0. \tag{7.12}$$

Then, we consider the expressions of $\mathbf{A}(\lambda;\mu)$, $\mathbf{D}(\lambda;\mu)$, $\mathbf{C}(\lambda;\mu)$ in terms of the entries of the fused Lax operators. For instance, if $N = 8$, we have (for conciseness we omit that the F's depend on the combination $\mu^{\frac{1}{2}}\lambda$, and the $\bar{F}$'s on the combination $\mu^{\frac{1}{2}}/\lambda$):

$$\begin{aligned}
\mathbf{A}(\lambda;\mu) &= \left[\bar{F}_{11}^{(8,7)}F_{11}^{(6,5)} + \bar{F}_{12}^{(8,7)}F_{21}^{(6,5)}\right]\left[\bar{F}_{11}^{(4,3)}F_{11}^{(2,1)} + \bar{F}_{12}^{(4,3)}F_{21}^{(2,1)}\right]\\
&\quad + \left[\bar{F}_{11}^{(8,7)}F_{12}^{(6,5)} + \bar{F}_{12}^{(8,7)}F_{22}^{(6,5)}\right]\left[\bar{F}_{21}^{(4,3)}F_{11}^{(2,1)} + \bar{F}_{22}^{(4,3)}F_{21}^{(2,1)}\right]\\
\mathbf{D}(\lambda;\mu) &= \left[\bar{F}_{21}^{(8,7)}F_{11}^{(6,5)} + \bar{F}_{22}^{(8,7)}F_{21}^{(6,5)}\right]\left[\bar{F}_{11}^{(4,3)}F_{12}^{(2,1)} + \bar{F}_{12}^{(4,3)}F_{22}^{(2,1)}\right]\\
&\quad + \left[\bar{F}_{21}^{(8,7)}F_{12}^{(6,5)} + \bar{F}_{22}^{(8,7)}F_{22}^{(6,5)}\right]\left[\bar{F}_{21}^{(4,3)}F_{12}^{(2,1)} + \bar{F}_{22}^{(4,3)}F_{22}^{(2,1)}\right]\\
\mathbf{C}(\lambda;\mu) &= \left[\bar{F}_{21}^{(8,7)}F_{11}^{(6,5)} + \bar{F}_{22}^{(8,7)}F_{21}^{(6,5)}\right]\left[\bar{F}_{11}^{(4,3)}F_{11}^{(2,1)} + \bar{F}_{12}^{(4,3)}F_{21}^{(2,1)}\right]\\
&\quad + \left[\bar{F}_{21}^{(8,7)}F_{12}^{(6,5)} + \bar{F}_{22}^{(8,7)}F_{22}^{(6,5)}\right]\left[\bar{F}_{21}^{(4,3)}F_{11}^{(2,1)} + \bar{F}_{22}^{(4,3)}F_{21}^{(2,1)}\right].
\end{aligned} \tag{7.13}$$

Hence, it is crucial that $F_{ij}^{(k,k-1)}(\lambda)$ and $\bar{F}_{i'j'}^{(k',k'-1)}(\lambda')$ $(k,k' \in 2\mathbb{N};\ 2 \leqslant k,k' \leqslant N)$—not necessarily in a representation—satisfy, in addition to the previous equations (6.16)–(6.19) and their right counterparts—obtained from (6.16)–(6.19) by replacing q with q^{-1}—mixed exchange relations, following directly from the W's exchange algebra:

- exchange (21)–(11)

$$\begin{aligned}
F_{21}^{(4i+2,4i+1)}(\lambda)\bar{F}_{11}^{(4i,4i-1)}(\lambda') &= q^{\frac{1}{2}}\bar{F}_{11}^{(4i,4i-1)}(\lambda')F_{21}^{(4i+2,4i+1)}(\lambda)\\
\bar{F}_{21}^{(4i,4i-1)}(\lambda')F_{11}^{(4i-2,4i-3)}(\lambda) &= q^{-\frac{1}{2}}F_{11}^{(4i-2,4i-3)}(\lambda)\bar{F}_{21}^{(4i,4i-1)}(\lambda')\\
\bar{F}_{21}^{(N,N-1)}(\lambda')F_{11}^{(2,1)}(\lambda) &= q^{\frac{1}{2}}F_{11}^{(2,1)}(\lambda)\bar{F}_{21}^{(N,N-1)}(\lambda')
\end{aligned} \tag{7.14}$$

- exchange (21)–(12)

$$\begin{aligned}
F_{21}^{(4i+2,4i+1)}(\lambda)\bar{F}_{12}^{(4i,4i-1)}(\lambda') &= q^{\frac{1}{2}}\bar{F}_{12}^{(4i,4i-1)}(\lambda')F_{21}^{(4i+2,4i+1)}(\lambda)\\
\bar{F}_{21}^{(4i,4i-1)}(\lambda')F_{12}^{(4i-2,4i-3)}(\lambda) &= q^{-\frac{1}{2}}F_{12}^{(4i-2,4i-3)}(\lambda)\bar{F}_{21}^{(4i,4i-1)}(\lambda')\\
\bar{F}_{21}^{(N,N-1)}(\lambda')F_{12}^{(2,1)}(\lambda) &= q^{-\frac{1}{2}}F_{12}^{(2,1)}(\lambda)\bar{F}_{21}^{(N,N-1)}(\lambda')
\end{aligned} \tag{7.15}$$

- exchange (21)–(22)

$$\begin{aligned}
F_{21}^{(4i+2,4i+1)}(\lambda)\bar{F}_{22}^{(4i,4i-1)}(\lambda') &= q^{-\frac{1}{2}}\bar{F}_{22}^{(4i,4i-1)}(\lambda')F_{21}^{(4i+2,4i+1)}(\lambda)\\
\bar{F}_{21}^{(4i,4i-1)}(\lambda')F_{22}^{(4i-2,4i-3)}(\lambda) &= q^{\frac{1}{2}}F_{22}^{(4i-2,4i-3)}(\lambda)\bar{F}_{21}^{(4i,4i-1)}(\lambda')\\
\bar{F}_{21}^{(N,N-1)}(\lambda')F_{22}^{(2,1)}(\lambda) &= q^{-\frac{1}{2}}F_{22}^{(2,1)}(\lambda)\bar{F}_{21}^{(N,N-1)}(\lambda')
\end{aligned} \tag{7.16}$$

- commutation if $2 < |k-k'| < N-2$

$$\left[\bar{F}_{ij}^{(k,k-1)}(\lambda),\ F_{i'j'}^{(k',k'-1)}(\lambda')\right] = 0. \tag{7.17}$$

Indeed, after iterated use of the exchange properties (7.14)–(7.17), we can accumulate all the factors $\bar{F}_{21}^{(4i,4i-1)}$, $F_{21}^{(4i-2,4i-3)}$ to the right of the addenda in expressions of **A**, **D**, **C**. From the form of these (see, e.g., formulae (7.13) in the case $N = 8$) and from the annihilation properties (7.12) it then follows:

$$\begin{aligned}
\mathbf{A}(\lambda;\mu)\mathbf{\Omega} &= \prod_{i=1}^{\overleftarrow{N/4}}\bar{F}_{11}^{(4i,4i-1)}\left(\frac{\mu^{1/2}}{\lambda}\right)F_{11}^{(4i-2,4i-3)}\left(\mu^{1/2}\lambda\right)\mathbf{\Omega}\\
\mathbf{D}(\lambda;\mu)\mathbf{\Omega} &= \prod_{i=1}^{\overleftarrow{N/4}}\bar{F}_{22}^{(4i,4i-1)}\left(\frac{\mu^{1/2}}{\lambda}\right)F_{22}^{(4i-2,4i-3)}\left(\mu^{1/2}\lambda\right)\mathbf{\Omega}\\
\mathbf{C}(\lambda;\mu)\mathbf{\Omega} &= 0.
\end{aligned} \tag{7.18}$$

We have already proved part of the statement in (7.18) and we complete it through finding the eigenvalues of $\mathbf{A}$ and $\mathbf{D}$ over $\mathbf{\Omega}$ in the following theorem and corollary.

Theorem 3. *The action of the ordered products of the operators* $\mathbf{F}^i_{11}$ *and* $\mathbf{F}^i_{22}$, *defined by*

$$\mathbf{F}^i_{11}(\lambda;\mu) \equiv \bar{F}^{(4i,4i-1)}_{11}\left(\frac{\mu^{1/2}}{\lambda}\right) F^{(4i-2,4i-3)}_{11}\left(\mu^{1/2}\lambda\right) \tag{7.19}$$

$$\mathbf{F}^i_{22}(\lambda;\mu) \equiv \bar{F}^{(4i,4i-1)}_{22}\left(\frac{\mu^{1/2}}{\lambda}\right) F^{(4i-2,4i-3)}_{22}\left(\mu^{1/2}\lambda\right) \tag{7.20}$$

on the states (7.11) is the following ($1 \leqslant i \leqslant N/4$):

$$\left[\prod_{j=1}^{\overset{i}{\leftarrow}} \mathbf{F}^j_{11}(\lambda;\mu)\mathbf{\Omega}\right](x_1,\ldots,x_N) = q^{-\frac{1}{2}} \exp\left(-\mathrm{i}\left(x_{4i+1} + 2\sum_{j=1}^{2i-1}(-)^j x_{2j+1} + x_1\right)\right)$$
$$\times(1-\Delta^2\mu\lambda^2 q^{-1})^i \left(1-\Delta^2\frac{\mu}{\lambda^2}q\right)^i \mathbf{\Omega}(x_1,\ldots,x_N)$$

$$\left[\prod_{j=1}^{\overset{i}{\leftarrow}} \mathbf{F}^j_{22}(\lambda;\mu)\mathbf{\Omega}\right](x_1,\ldots,x_N) = q^{-\frac{1}{2}} \exp\left(\mathrm{i}\left(x_{4i+1} + 2\sum_{j=1}^{2i-1}(-)^j x_{2j+1} + x_1\right)\right)$$
$$\times(1-\Delta^2\mu\lambda^2 q)^i \left(1-\Delta^2\frac{\mu}{\lambda^2}q^{-1}\right)^i \mathbf{\Omega}(x_1,\ldots,x_N).$$

Proof. We show by induction the first formula. For $i = 1$ we have, using formulae (6.7), (6.42)

$$\begin{aligned}\left[\mathbf{F}^1_{11}(\lambda;\mu)\mathbf{\Omega}\right](x_1,\ldots,x_N) &= \mathrm{e}^{-\mathrm{i}\left(x_5-2x_3+x_1-\frac{\pi\beta^2}{2}\right)}\mathbf{\Omega}(x_1^+,x_2^+,x_3^+,x_4^+,x_5,\ldots,x_N)\\ &+\Delta^2\mu\lambda^2 q^{-1}\,\mathrm{e}^{-\mathrm{i}\left(x_5-2x_3+2x_2-x_1-\frac{\pi\beta^2}{2}\right)}\mathbf{\Omega}(x_1^+,x_2^-,x_3^+,x_4^+,x_5,\ldots,x_N)\\ &+\Delta^2\frac{\mu}{\lambda^2}q\,\mathrm{e}^{-\mathrm{i}\left(x_5-2x_4+x_1-\frac{\pi\beta^2}{2}\right)}\mathbf{\Omega}(x_1^+,x_2^+,x_3^+,x_4^-,x_5,\ldots,x_N)\\ &+\Delta^4\mu^2\,\mathrm{e}^{-\mathrm{i}\left(x_5-2x_4+2x_2-x_1-\frac{\pi\beta^2}{2}\right)}\mathbf{\Omega}(x_1^+,x_2^-,x_3^+,x_4^-,x_5,\ldots,x_N).\end{aligned} \tag{7.21}$$

Now we remark that $\mathbf{\Omega}(x_1^+,x_2^+,x_3^+,x_4^+,x_5,\ldots,x_N) = \mathbf{\Omega}(x_1,\ldots,x_N)$ and that the use of the shift property (6.12) for the function f contained in (7.11) gives

$$\mathbf{\Omega}(x_1^+,x_2^-,x_3^+,x_4^+,x_5,\ldots,x_N) = -\mathrm{e}^{-2\mathrm{i}(x_1-x_2)}\mathbf{\Omega}(x_1,\ldots,x_N)$$
$$\mathbf{\Omega}(x_1^+,x_2^+,x_3^+,x_4^-,x_5,\ldots,x_N) = -\mathrm{e}^{2\mathrm{i}(x_3-x_4)}\mathbf{\Omega}(x_1,\ldots,x_N)$$
$$\mathbf{\Omega}(x_1^+,x_2^-,x_3^+,x_4^-,x_5,\ldots,x_N) = \mathrm{e}^{-2\mathrm{i}(x_1-x_2)}\,\mathrm{e}^{2\mathrm{i}(x_3-x_4)}\mathbf{\Omega}(x_1,\ldots,x_N)$$

because the shifts in the variables $x_1,\ldots,x_4$ do not affect the delta function contained in (7.11). Therefore, all the terms in (7.21) are proportional and the final result is

$$\left[\mathbf{F}^1_{11}(\lambda;\mu)\mathbf{\Omega}\right](x_1,\ldots,x_N) = q^{-\frac{1}{2}}\,\mathrm{e}^{-\mathrm{i}(x_5-2x_3+x_1)}\left(1-\frac{\Delta^2\mu\lambda^2}{q}\right)\left(1-\frac{\Delta^2\mu q}{\lambda^2}\right)\mathbf{\Omega}(x_1,\ldots,x_N),$$

which is the first formula of theorem 3 for $i = 1$.

For $2 \leqslant i \leqslant N/4$ we have from (6.7), (6.42)

$$\left[\prod_{j=1}^{\overleftarrow{i}} \mathbf{F}_{11}^{j}(\lambda;\mu)\boldsymbol{\Omega}\right](x_1,\ldots,x_N) = q^{-\frac{1}{2}}\,\mathrm{e}^{-\mathrm{i}x_{4i+1}}\Bigg\{\mathrm{e}^{\mathrm{i}(2x_{4i-1}-x_{4i-3})}$$

$$\times\left[\prod_{j=1}^{\overleftarrow{i-1}} \mathbf{F}_{11}^{j}(\lambda;\mu)\boldsymbol{\Omega}\right](x_1,\ldots,x_{4i-4},x_{4i-3}^{+},x_{4i-2}^{+},x_{4i-1}^{+},x_{4i}^{+},x_{4i+1},\ldots,x_N)$$

$$+\frac{\Delta^2\mu\lambda^2}{q}\,\mathrm{e}^{\mathrm{i}(2x_{4i-1}-2x_{4i-2}+x_{4i-3})}\left[\prod_{j=1}^{\overleftarrow{i-1}} \mathbf{F}_{11}^{j}(\lambda;\mu)\boldsymbol{\Omega}\right]$$

$$\times(x_1,\ldots,x_{4i-4},x_{4i-3}^{+},x_{4i-2}^{-},x_{4i-1}^{+},x_{4i}^{+},x_{4i+1},\ldots,x_N)$$

$$+\frac{\Delta^2\mu q}{\lambda^2}\,\mathrm{e}^{\mathrm{i}(2x_{4i}-x_{4i-3})}\left[\prod_{j=1}^{\overleftarrow{i-1}} \mathbf{F}_{11}^{j}(\lambda;\mu)\boldsymbol{\Omega}\right]$$

$$\times(x_1,\ldots,x_{4i-4},x_{4i-3}^{+},x_{4i-2}^{+},x_{4i-1}^{+},x_{4i}^{-},x_{4i+1},\ldots,x_N)$$

$$+\Delta^4\mu^2\,\mathrm{e}^{\mathrm{i}(2x_{4i}-2x_{4i-2}+x_{4i-3})}\left[\prod_{j=1}^{\overleftarrow{i-1}} \mathbf{F}_{11}^{j}(\lambda;\mu)\boldsymbol{\Omega}\right]$$

$$\times(x_1,\ldots,x_{4i-4},x_{4i-3}^{+},x_{4i-2}^{-},x_{4i-1}^{+},x_{4i}^{-},x_{4i+1},\ldots,x_N)\Bigg\}.$$

Using the inductive hypothesis, we get

$$\left[\prod_{j=1}^{\overleftarrow{i}} \mathbf{F}_{11}^{j}(\lambda;\mu)\boldsymbol{\Omega}\right](x_1,\ldots,x_N) = q^{-1}\Bigg\{\mathrm{e}^{-\mathrm{i}(x_{4i+1}-2x_{4i-1}+x_{4i-3})}$$

$$\times\exp\left(-\mathrm{i}\left(x_{4i-3}^{+}+2\sum_{j=1}^{2i-3}(-)^j x_{2j+1}+x_1\right)\right)(1-\Delta^2\mu\lambda^2q^{-1})^{i-1}$$

$$\times\left(1-\Delta^2\frac{\mu}{\lambda^2}q\right)^{\mathrm{i}-1}\boldsymbol{\Omega}(x_1,\ldots,x_{4i-4},x_{4i-3}^{+},x_{4i-2}^{+},x_{4i-1}^{+},x_{4i}^{+},x_{4i+1},\ldots,x_N)$$

$$+\frac{\Delta^2\mu\lambda^2}{q}\,\mathrm{e}^{-\mathrm{i}(x_{4i+1}-2x_{4i-1}+2x_{4i-2}-x_{4i-3})}\exp\left(-\mathrm{i}\left(x_{4i-3}^{+}+2\sum_{j=1}^{2i-3}(-)^j x_{2j+1}+x_1\right)\right)$$

$$\times(1-\Delta^2\mu\lambda^2q^{-1})^{i-1}\left(1-\Delta^2\frac{\mu}{\lambda^2}q\right)^{i-1}$$

$$\times\,\boldsymbol{\Omega}(x_1,\ldots,x_{4i-4},x_{4i-3}^{+},x_{4i-2}^{-},x_{4i-1}^{+},x_{4i}^{+},x_{4i+1},\ldots,x_N)$$

$$+\frac{\Delta^2\mu q}{\lambda^2}\,\mathrm{e}^{-\mathrm{i}(x_{4i+1}-2x_{4i}+x_{4i-3})}\exp\left(-\mathrm{i}\left(x_{4i-3}^{+}+2\sum_{j=1}^{2i-3}(-)^j x_{2j+1}+x_1\right)\right)$$

$$\times(1-\Delta^2\mu\lambda^2q^{-1})^{i-1}\left(1-\Delta^2\frac{\mu}{\lambda^2}q\right)^{i-1}$$

$$\times\, \mathbf{\Omega}(x_1,\ldots,x_{4i-4},x^+_{4i-3},x^+_{4i-2},x^+_{4i-1},x^-_{4i},x_{4i+1},\ldots,x_N)$$
$$+\,\Delta^4\mu^2\,\mathrm{e}^{-\mathrm{i}(x_{4i+1}-2x_{4i}+2x_{4i-2}-x_{4i-3})}\exp\left(-\mathrm{i}\left(x^+_{4i-3}+2\sum_{j=1}^{2i-3}(-)^j x_{2j+1}+x_1\right)\right)$$
$$\times\,(1-\Delta^2\mu\lambda^2q^{-1})^{i-1}\left(1-\Delta^2\frac{\mu}{\lambda^2}q\right)^{i-1}$$
$$\times\, \mathbf{\Omega}(x_1,\ldots,x_{4i-4},x^+_{4i-3},x^-_{4i-2},x^+_{4i-1},x^-_{4i},x_{4i+1},\ldots,x_N)\Bigg\}. \qquad (7.22)$$

As in the case $i=1$ we have

$$\mathbf{\Omega}(x_1,\ldots,x_{4i-4},x^+_{4i-3},x^+_{4i-2},x^+_{4i-1},x^+_{4i},x_{4i+1},\ldots,x_N)=\mathbf{\Omega}(x_1,\ldots,x_N) \qquad (7.23)$$

and the use of property (6.12) for the function f contained in (7.11) gives

$$\mathbf{\Omega}(x_1,\ldots,x_{4i-4},x^+_{4i-3},x^-_{4i-2},x^+_{4i-1},x^+_{4i},x_{4i+1},\ldots,x_N)=-\mathrm{e}^{-2\mathrm{i}(x_{4i-3}-x_{4i-2})}\mathbf{\Omega}(x_1,\ldots,x_N)$$
$$\mathbf{\Omega}(x_1,\ldots,x_{4i-4},x^+_{4i-3},x^+_{4i-2},x^+_{4i-1},x^-_{4i},x_{4i+1},\ldots,x_N)=-\mathrm{e}^{2\mathrm{i}(x_{4i-1}-x_{4i})}\mathbf{\Omega}(x_1,\ldots,x_N)$$
$$\mathbf{\Omega}(x_1,\ldots,x_{4i-4},x^+_{4i-3},x^-_{4i-2},x^+_{4i-1},x^-_{4i},x_{4i+1},\ldots,x_N)$$
$$=\mathrm{e}^{-2\mathrm{i}(x_{4i-3}-x_{4i-2})}\,\mathrm{e}^{2\mathrm{i}(x_{4i-1}-x_{4i})}\mathbf{\Omega}(x_1,\ldots,x_N)$$

because the shifts in the variables $x_{4i-3},\ldots,x_{4i}$ do not affect the delta function contained in (7.11). Hence, all the terms in (7.22) are proportional to $\mathbf{\Omega}$ and the final result is

$$\left[\prod_{j=1}^{\overleftarrow{i}}\mathbf{F}^j_{11}(\lambda;\mu)\mathbf{\Omega}\right](x_1,\ldots,x_N)=q^{-\frac{1}{2}}\exp\left(-\mathrm{i}\left(x_{4i+1}+2\sum_{j=1}^{2i-1}(-)^j x_{2j+1}+x_1\right)\right)$$
$$\times\,(1-\Delta^2\mu\lambda^2q^{-1})^{i}\left(1-\Delta^2\frac{\mu}{\lambda^2}q\right)^{i}\mathbf{\Omega}(x_1,\ldots,x_N) \qquad (7.24)$$

which is the first formula of theorem 3.

The proof for $\mathbf{F}_{22}$ elements follows the same lines and we do not write it. □

Corollary 3. *The states (7.11) are eigenvectors of the elements* $\mathbf{A}(\lambda;\mu)$ *and* $\mathbf{D}(\lambda;\mu)$ *of the monodromy matrix (7.1). The corresponding common eigenvalues are given by the following formulae:*

$$\mathbf{A}(\lambda;\mu)\mathbf{\Omega}=q^{-\frac{1}{2}}(1-\Delta^2\mu\lambda^2q^{-1})^{N/4}\left(1-\Delta^2\frac{\mu}{\lambda^2}q\right)^{N/4}\mathbf{\Omega} \qquad (7.25)$$
$$\mathbf{D}(\lambda;\mu)\mathbf{\Omega}=q^{-\frac{1}{2}}(1-\Delta^2\mu\lambda^2q)^{N/4}\left(1-\Delta^2\frac{\mu}{\lambda^2}q^{-1}\right)^{N/4}\mathbf{\Omega}. \qquad (7.26)$$

Proof. We apply theorem 3 for $i=N/4$ and note that the variable x_{N+1}, appearing in the formulae of theorem 3 for $i=N/4$, must be read as x_1. Hence, we have that the exponents in the second factors on the right-hand sides of the formulae of theorem 3 are proportional to $\sum_{i=1}^{N/4}(x_{4i-3}-x_{4i-1})$: therefore, they can be put equal to zero because of the delta function $\delta\big(\sum_{i=1}^{N/4}(x_{4i-3}-x_{4i-1})\big)$ in definition (7.11) of $\mathbf{\Omega}$. In this way we obtain formulae (7.25), (7.26). □

Eventually, formulae (7.18), (7.25), (7.26) show that (7.11) are pseudovacuum states for the monodromy matrix (7.1) with the same $\mathbf{A}(\lambda;\mu)$ and $\mathbf{D}(\lambda;\mu)$ eigenvalues, respectively, for any $f(x)$ solution of (6.12). Nevertheless, *a fortiori* the pseudovacua are non-ultralocal in this off-critical case.

In order to write down the BEs, we remark that from (4.4) it follows that monodromy matrix (7.1) satisfies the exchange relations (6.26) where Z_{ab}^{-1} is replaced by Z_{ab}. Hence, the braided exchange rules between $\mathbf{B}(\lambda';\mu)$ and $\mathbf{A}(\lambda;\mu)$, $\mathbf{D}(\lambda;\mu)$ are respectively (we suppress the dependence on μ for reasons of conciseness)

$$\mathbf{A}(\lambda)\mathbf{B}(\lambda') = \frac{q}{a\left(\frac{\lambda'}{\lambda}\right)}\mathbf{B}(\lambda')\mathbf{A}(\lambda) - q\frac{b\left(\frac{\lambda'}{\lambda}\right)}{a\left(\frac{\lambda'}{\lambda}\right)}\mathbf{B}(\lambda)\mathbf{A}(\lambda') \tag{7.27}$$

$$\mathbf{D}(\lambda)\mathbf{B}(\lambda') = \frac{q^{-1}}{a\left(\frac{\lambda}{\lambda'}\right)}\mathbf{B}(\lambda')\mathbf{D}(\lambda) - q^{-1}\frac{b\left(\frac{\lambda}{\lambda'}\right)}{a\left(\frac{\lambda}{\lambda'}\right)}\mathbf{B}(\lambda)\mathbf{D}(\lambda'). \tag{7.28}$$

As in the previous section, the states

$$\mathbf{\Psi}(\lambda_1,\ldots,\lambda_l) = \prod_{r=1}^{l}\mathbf{B}(\lambda_r)\mathbf{\Omega} \tag{7.29}$$

are eigenstates of the transfer matrix $\mathbf{T}(\lambda) = \mathbf{A}(\lambda) + \mathbf{D}(\lambda)$ (Bethe states) only if the set of complex numbers $\{\lambda_1,\ldots,\lambda_l\}$ (Bethe roots) satisfies the following BEs:

$$q^{2l}\prod_{\substack{r=1\\ r\neq s}}^{l}\frac{q\lambda_r^2 - q^{-1}\lambda_s^2}{q^{-1}\lambda_r^2 - q\lambda_s^2} = \left[\frac{\left(1-\Delta^2\mu\lambda_s^2 q\right)\left(1-\Delta^2\frac{\mu}{\lambda_s^2}q^{-1}\right)}{\left(1-\Delta^2\mu\lambda_s^2 q^{-1}\right)\left(1-\Delta^2\frac{\mu}{\lambda_s^2}q\right)}\right]^{N/4}. \tag{7.30}$$

It is useful to rewrite (7.30) in trigonometric form. Let us define the new variables Θ, α and α_r:

$$\Delta^2\mu \equiv \mathrm{e}^{-2\Theta} \qquad \lambda \equiv \mathrm{e}^{\alpha} \qquad \lambda_r \equiv \mathrm{e}^{\alpha_r}. \tag{7.31}$$

In terms of these variables the BEs (7.30) are $\left(q = \mathrm{e}^{-\mathrm{i}\pi\beta^2}\right)$

$$\mathrm{e}^{-2\mathrm{i}\pi\beta^2 l}\prod_{\substack{r=1\\ r\neq s}}^{l}\frac{\sinh\left(\alpha_s-\alpha_r+\mathrm{i}\pi\beta^2\right)}{\sinh\left(\alpha_s-\alpha_r-\mathrm{i}\pi\beta^2\right)} = \left[\frac{\sinh\left(\alpha_s+\Theta-\frac{\mathrm{i}\pi\beta^2}{2}\right)\sinh\left(\alpha_s-\Theta-\frac{\mathrm{i}\pi\beta^2}{2}\right)}{\sinh\left(\alpha_s+\Theta+\frac{\mathrm{i}\pi\beta^2}{2}\right)\sinh\left(\alpha_s-\Theta+\frac{\mathrm{i}\pi\beta^2}{2}\right)}\right]^{N/4}. \tag{7.32}$$

Finally, from equations (7.27), (7.28) and from (7.25) and (7.26) it follows that the eigenvalues $\mathbf{\Lambda}(\lambda,\{\lambda_r\})$ of the transfer matrix $\mathbf{T}(\lambda)$ on the Bethe states (7.29), (7.30) are

$$\begin{aligned}\mathbf{\Lambda}(\lambda,\{\lambda_r\}) = {}& q^l\prod_{r=1}^{l}\frac{q^{-1}\lambda^2 - q\lambda_r^2}{\lambda^2-\lambda_r^2}q^{-\frac{1}{2}}[(1-\Delta^2\mu\lambda^2q^{-1})(1-\Delta^2\mu\lambda^{-2}q)]^{N/4}\\ &+ q^{-l}\prod_{r=1}^{l}\frac{q\lambda^2-q^{-1}\lambda_r^2}{\lambda^2-\lambda_r^2}q^{-\frac{1}{2}}[(1-\Delta^2\mu\lambda^2q)(1-\Delta^2\mu\lambda^{-2}q^{-1})]^{N/4}.\end{aligned} \tag{7.33}$$

In addition, it is useful to write also the eigenvalues of the transfer matrix in trigonometric form. After inserting (7.31) in (7.33), we obtain

$$\begin{aligned}\mathrm{e}^{-\frac{\mathrm{i}\pi\beta^2}{2}+\frac{\Theta N}{2}}\mathbf{\Lambda}(\alpha,\{\alpha_r\}) = {}& \mathrm{e}^{-\mathrm{i}\pi\beta^2 l}\prod_{r=1}^{l}\frac{\sinh\left(\alpha-\alpha_r+\mathrm{i}\pi\beta^2\right)}{\sinh(\alpha-\alpha_r)}\left[4\sinh\left(\Theta-\alpha-\frac{\mathrm{i}\pi\beta^2}{2}\right)\right.\\ &\times\left.\sinh\left(\Theta+\alpha+\frac{\mathrm{i}\pi\beta^2}{2}\right)\right]^{N/4} + \mathrm{e}^{\mathrm{i}\pi\beta^2 l}\prod_{r=1}^{l}\frac{\sinh\left(\alpha-\alpha_r-\mathrm{i}\pi\beta^2\right)}{\sinh(\alpha-\alpha_r)}\\ &\times\left[4\sinh\left(\Theta-\alpha+\frac{\mathrm{i}\pi\beta^2}{2}\right)\sinh\left(\Theta+\alpha-\frac{\mathrm{i}\pi\beta^2}{2}\right)\right]^{N/4}.\end{aligned} \tag{7.34}$$

Now, for completeness, we illustrate the main results regarding the other choice of off-critical monodromy matrix (4.25). The calculations for obtaining

- the pseudovacua,
- the Bethe states and the Bethe equations and
- the eigenvalues of the transfer matrix

have been carried out in a way parallel to that performed in case (4.24). In what follows, we summarize only the final results.

The pseudovacua in the coordinate representation are given by

$$\mathbf{\Omega}'(x_1,\ldots,x_N)=\prod_{i=1}^{N/4}f(x_{4i-1}-x_{4i})f(x_{4i-3}-x_{4i-2})^{-1}\delta\left(\sum_{i=1}^{N/4}(x_{4i-3}-x_{4i-1})\right). \tag{7.35}$$

The Bethe states are

$$\mathbf{\Psi}'(\lambda_1,\ldots,\lambda_l)=\prod_{r=1}^{l}\mathbf{B}'(\lambda_r)\mathbf{\Omega}', \tag{7.36}$$

in addition to the BEs

$$q^{-2l}\prod_{\substack{r=1\\r\neq s}}^{l}\frac{q\lambda_r^2-q^{-1}\lambda_s^2}{q^{-1}\lambda_r^2-q\lambda_s^2}=\left[\frac{\left(1-\Delta^2\mu\lambda_s^2q\right)\left(1-\Delta^2\frac{\mu}{\lambda_s^2}q^{-1}\right)}{\left(1-\Delta^2\mu\lambda_s^2q^{-1}\right)\left(1-\Delta^2\frac{\mu}{\lambda_s^2}q\right)}\right]^{N/4} \tag{7.37}$$

or in trigonometric form

$$\mathrm{e}^{2\mathrm{i}\pi\beta^2l}\prod_{\substack{r=1\\r\neq s}}^{l}\frac{\sinh\left(\alpha_s-\alpha_r+\mathrm{i}\pi\beta^2\right)}{\sinh\left(\alpha_s-\alpha_r-\mathrm{i}\pi\beta^2\right)}=\left[\frac{\sinh\left(\alpha_s+\Theta-\frac{\mathrm{i}\pi\beta^2}{2}\right)\sinh\left(\alpha_s-\Theta-\frac{\mathrm{i}\pi\beta^2}{2}\right)}{\sinh\left(\alpha_s+\Theta+\frac{\mathrm{i}\pi\beta^2}{2}\right)\sinh\left(\alpha_s-\Theta+\frac{\mathrm{i}\pi\beta^2}{2}\right)}\right]^{N/4}. \tag{7.38}$$

The eigenvalues of the transfer matrix are

$$\begin{aligned}\mathbf{\Lambda}'(\lambda,\{\lambda_r\})={}&q^{-l}\prod_{r=1}^{l}\frac{q^{-1}\lambda^2-q\lambda_r^2}{\lambda^2-\lambda_r^2}q^{\frac12}[(1-\Delta^2\mu\lambda^2q^{-1})(1-\Delta^2\mu\lambda^{-2}q)]^{N/4}\\&+q^{l}\prod_{r=1}^{l}\frac{q\lambda^2-q^{-1}\lambda_r^2}{\lambda^2-\lambda_r^2}q^{\frac12}[(1-\Delta^2\mu\lambda^2q)(1-\Delta^2\mu\lambda^{-2}q^{-1})]^{N/4}\end{aligned} \tag{7.39}$$

or in trigonometric form

$$\begin{aligned}\mathrm{e}^{\frac{\mathrm{i}\pi\beta^2}{2}+\frac{\Theta N}{2}}\mathbf{\Lambda}'(\alpha,\{\alpha_r\})={}&\mathrm{e}^{\mathrm{i}\pi\beta^2l}\prod_{r=1}^{l}\frac{\sinh\left(\alpha-\alpha_r+\mathrm{i}\pi\beta^2\right)}{\sinh(\alpha-\alpha_r)}\left[4\sinh\left(\Theta-\alpha-\frac{\mathrm{i}\pi\beta^2}{2}\right)\right.\\&\times\left.\sinh\left(\Theta+\alpha+\frac{\mathrm{i}\pi\beta^2}{2}\right)\right]^{N/4}+\mathrm{e}^{-\mathrm{i}\pi\beta^2l}\prod_{r=1}^{l}\frac{\sinh(\alpha-\alpha_r-\mathrm{i}\pi\beta^2)}{\sinh(\alpha-\alpha_r)}\\&\times\left[4\sinh\left(\Theta-\alpha+\frac{\mathrm{i}\pi\beta^2}{2}\right)\sinh\left(\Theta+\alpha-\frac{\mathrm{i}\pi\beta^2}{2}\right)\right]^{N/4}.\end{aligned} \tag{7.40}$$

In this section we have calculated the eigenvalues of the two lattice transfer matrices associated with the monodromy matrices (4.24) and (4.25). We will show in the following section that these eigenvalues in the *limit* $\mu \to 0$ reduce to the conformal right and left ones, respectively. This reinforces our idea that the monodromy matrices (4.24) and (4.25) will describe, after the (cylinder) continuum limit, *a sort of perturbation* from CFT. We will discuss the nature of these theories rigorously in a future paper [29].

8. Conformal limits of the off-critical transfer matrix eigenvalues

In this section we show that, after suitable rescaling of the spectral parameter and the Bethe roots, in the limit $\mu \to 0$, the eigenvalues of the off-critical transfer matrices (7.33) and (7.39) are proportional respectively to the eigenvalues of the right and left conformal transfer matrices (6.49) and (6.39).

Indeed, let us consider the eigenvalue (7.33) and calculate the limit

$$\lim_{\mu\to 0} \mathbf{\Lambda}\big(\lambda\mu^{1/2}, \{\lambda_r\mu^{1/2}\}\big) = q^l \prod_{r=1}^{l} \frac{q^{-1}\lambda^2 - q\lambda_r^2}{\lambda^2 - \lambda_r^2} q^{-\frac{1}{2}}(1 - \Delta^2\lambda^{-2}q)^{N/4} + q^{-l} \prod_{r=1}^{l} \frac{q\lambda^2 - q^{-1}\lambda_r^2}{\lambda^2 - \lambda_r^2} q^{-\frac{1}{2}}(1 - \Delta^2\lambda^{-2}q^{-1})^{N/4}. \tag{8.1}$$

The parameters λ_r contained in this relation must satisfy a system of Bethe equations which is obtained from (7.30) by rescaling $\lambda_r \to \lambda_r\mu^{1/2}$ and taking the limit $\mu \to 0$. The equations obtained in such a way are the Bethe equations (6.47) for the right conformal theory, where N is replaced by $N/2$. Therefore, the rhs of (8.1) as a function of λ is proportional, by the factor $q^{-N/8}$, to the right conformal eigenvalue (6.49), where N is replaced by $N/2$:

$$\begin{aligned} \lim_{\mu\to 0} \mathbf{\Lambda}\big(\lambda\mu^{1/2}, \{\lambda_r\mu^{1/2}\}\big) &= q^l \prod_{r=1}^{l} \frac{q^{-1}\lambda^2 - q\lambda_r^2}{\lambda^2 - \lambda_r^2} q^{-\frac{1}{2}}(1 - \Delta^2\lambda^{-2}q)^{N/4} \\ &\quad + q^{-l} \prod_{r=1}^{l} \frac{q\lambda^2 - q^{-1}\lambda_r^2}{\lambda^2 - \lambda_r^2} q^{-\frac{1}{2}}(1 - \Delta^2\lambda^{-2}q^{-1})^{N/4} \\ &= q^{-N/8}\Bigg\{ q^l \prod_{r=1}^{l} \frac{q^{-1}\lambda^2 - q\lambda_r^2}{\lambda^2 - \lambda_r^2} q^{\frac{1}{2}(\frac{N}{4}-1)}(1 - \Delta^2\lambda^{-2}q)^{N/4} \\ &\quad + q^{-l} \prod_{r=1}^{l} \frac{q\lambda^2 - q^{-1}\lambda_r^2}{\lambda^2 - \lambda_r^2} q^{\frac{1}{2}(\frac{N}{4}-1)}(1 - \Delta^2\lambda^{-2}q^{-1})^{N/4}\Bigg\}. \end{aligned} \tag{8.2}$$

Let us now consider eigenvalue (7.39) and perform the following limit:

$$\lim_{\mu\to 0} \mathbf{\Lambda}'\big(\lambda\mu^{-1/2}, \{\lambda_r\mu^{-1/2}\}\big) = q^{-l} \prod_{r=1}^{l} \frac{q^{-1}\lambda^2 - q\lambda_r^2}{\lambda^2 - \lambda_r^2} q^{\frac{1}{2}}(1 - \Delta^2\lambda^{2}q^{-1})^{N/4} + q^{l} \prod_{r=1}^{l} \frac{q\lambda^2 - q^{-1}\lambda_r^2}{\lambda^2 - \lambda_r^2} q^{\frac{1}{2}}(1 - \Delta^2\lambda^{2}q)^{N/4}. \tag{8.3}$$

The parameters λ_r contained in this relation must satisfy a system of Bethe equations which is obtained from (7.37) by rescaling λ_r into $\lambda_r\mu^{-1/2}$ and by taking the limit $\mu \to 0$. These equations are the Bethe equations for left conformal theories (6.35), where N is replaced by $N/2$. Hence, the rhs of (8.3) is proportional, by a factor $q^{N/8}$, to the left conformal eigenvalue (6.39) with N replaced by $N/2$:

$$\lim_{\mu\to 0}\mathbf{\Lambda}'\big(\lambda\mu^{-1/2},\{\lambda_r\mu^{-1/2}\}\big)=q^{-l}\prod_{r=1}^{l}\frac{q^{-1}\lambda^2-q\lambda_r^2}{\lambda^2-\lambda_r^2}q^{\frac{1}{2}}(1-\Delta^2\lambda^2q^{-1})^{N/4}$$
$$+q^{l}\prod_{r=1}^{l}\frac{q\lambda^2-q^{-1}\lambda_r^2}{\lambda^2-\lambda_r^2}q^{\frac{1}{2}}(1-\Delta^2\lambda^2q)^{N/4}$$
$$=q^{N/8}\left\{q^{-l}\prod_{r=1}^{l}\frac{q^{-1}\lambda^2-q\lambda_r^2}{\lambda^2-\lambda_r^2}q^{-\frac{1}{2}(\frac{N}{4}-1)}(1-\Delta^2\lambda^2q^{-1})^{N/4}\right.$$
$$\left.+\,q^{l}\prod_{r=1}^{l}\frac{q\lambda^2-q^{-1}\lambda_r^2}{\lambda^2-\lambda_r^2}q^{-\frac{1}{2}(\frac{N}{4}-1)}(1-\Delta^2\lambda^2q)^{N/4}\right\}. \tag{8.4}$$

9. Cylinder scaling limits

In this section we derive the scaling expressions for the critical and off-critical monodromy matrices (4.22)–(4.25) in the cylinder limit defined by

$$N\to\infty \quad \text{and fixed} \quad R\equiv N\Delta. \tag{9.1}$$

The previous limit (9.1) will be taken in a rigorous way in a forthcoming paper [29] defining in this way the continuum cylinder limit, whereas now we illustrate here a heuristic operatorial limit to gain further clues about the physical meaning of the monodromy matrices previously analysed. However, we believe that the results we will show are substantially correct [29].

From the definitions of $V_m^{\pm}$ (3.12), (3.13) one obtains immediately that their behaviour in the cylinder scaling limit is

$$V_m^-=-\Delta\phi'(y_{2m})+O(\Delta^2) \qquad V_m^+=-\Delta\bar{\phi}'(\bar{y}_{2m})+O(\Delta^2) \tag{9.2}$$

where $y_{2m}=\bar{y}_{2m}=m\frac{R}{N}$. Hence, in this limit the Lax operators (3.19) behave as follows:

$$L_m(\lambda)=1+\Delta\mathcal{L}\left(m\frac{R}{N},\lambda\right)+O(\Delta^2) \qquad \bar{L}_m(\lambda^{-1})=1+\Delta\bar{\mathcal{L}}\left(m\frac{R}{N},\lambda^{-1}\right)+O(\Delta^2) \tag{9.3}$$

where we have defined

$$\mathcal{L}(y,\lambda)\equiv\begin{pmatrix}\mathrm{i}\phi'(y) & \lambda\\ \lambda & -\mathrm{i}\phi'(y)\end{pmatrix} \qquad \bar{\mathcal{L}}(\bar{y},\lambda^{-1})\equiv\begin{pmatrix}\mathrm{i}\bar{\phi}'(\bar{y}) & \lambda^{-1}\\ \lambda^{-1} & -\mathrm{i}\bar{\phi}'(\bar{y})\end{pmatrix}. \tag{9.4}$$

Finally, by using (9.3) we have that the left (4.22) and right (4.23) monodromy matrices assume in the cylinder scaling limit the form

$$M(\lambda)=\prod_{k=1}^{\overleftarrow{N}}\left[1+\Delta\mathcal{L}\left(k\frac{R}{N},\lambda\right)+O(\Delta^2)\right]\to\mathcal{P}\exp\int_0^R \mathrm{d}y\,\mathcal{L}(y,\lambda) \tag{9.5}$$

$$\bar{M}(\lambda)=\prod_{k=1}^{\overleftarrow{N}}\left[1+\Delta\bar{\mathcal{L}}\left(k\frac{R}{N},\lambda^{-1}\right)+O(\Delta^2)\right]\to\mathcal{P}\exp\int_0^R \mathrm{d}\bar{y}\,\bar{\mathcal{L}}(\bar{y},\lambda^{-1}). \tag{9.6}$$

At this point it is important to observe the slight difference between the limit expressions (9.5), (9.6) and the chiral and anti-chiral monodromy matrices proposed in [27]. Indeed, writing formulae (9.4) in the following way:

$$\mathcal{L}(y,\lambda)=\mathrm{i}\phi'(y)H+\lambda(E+F) \qquad \bar{\mathcal{L}}(y,\lambda)=\mathrm{i}\bar{\phi}'(y)H+\lambda(E+F) \tag{9.7}$$

where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{9.8}$$

we finally obtain these expressions for (9.5) and (9.6), respectively,

$$M(\lambda) = \mathcal{P} \exp \int_0^R \mathrm{d}y\, [\mathrm{i}\phi'(y)H + \lambda(E+F)] \tag{9.9}$$

and

$$\bar{M}(\lambda) = \mathcal{P} \exp \int_0^R \mathrm{d}\bar{y}\, [\mathrm{i}\bar{\phi}'(\bar{y})H + \lambda^{-1}(E+F)]. \tag{9.10}$$

We will show in a forthcoming paper [29] how to reproduce, starting from a regularized expression on a lattice, the chiral and anti-chiral monodromy matrices of [27] and why these verify the Yang–Baxter algebra instead of our braided version.

Let us now derive the expressions for the monodromy matrices (4.24)–(4.25) in the cylinder scaling limit. For what concerns the monodromy matrix (4.24) we have

$$\begin{aligned}
\mathbf{M}(\lambda) &= \overleftarrow{\prod_{i=1}^{N/4}} \left[1 + \Delta\bar{\mathcal{L}}\left(\frac{4i}{N}R, \frac{\mu^{1/2}}{\lambda}\right) + O(\Delta^2)\right]\left[1 + \Delta\bar{\mathcal{L}}\left(\frac{4i-1}{N}R, \frac{\mu^{1/2}}{\lambda}\right) + O(\Delta^2)\right] \\
&\quad \times \left[1 + \Delta\mathcal{L}\left(\frac{4i-2}{N}R, \mu^{1/2}\lambda\right) + O(\Delta^2)\right] \\
&\quad \times \left[1 + \Delta\mathcal{L}\left(\frac{4i-3}{N}R, \mu^{1/2}\lambda\right) + O(\Delta^2)\right] \\
&= \overleftarrow{\prod_{i=1}^{N/4}} \left[1 + \Delta\bar{\mathcal{L}}\left(\frac{4i}{N}R, \frac{\mu^{1/2}}{\lambda}\right) + \Delta\bar{\mathcal{L}}\left(\frac{4i-1}{N}R, \frac{\mu^{1/2}}{\lambda}\right)\right. \\
&\quad \left. + \Delta\mathcal{L}\left(\frac{4i-2}{N}R, \mu^{1/2}\lambda\right) + \Delta\mathcal{L}\left(\frac{4i-3}{N}R, \mu^{1/2}\lambda\right) + O(\Delta^2)\right] \\
&\quad \to \mathcal{P} \exp \frac{1}{2}\int_0^R \mathrm{d}y \left[\bar{\mathcal{L}}\left(y, \frac{\mu^{1/2}}{\lambda}\right) + \mathcal{L}\big(y, \mu^{1/2}\lambda\big)\right] \equiv \mathcal{M}(\lambda).
\end{aligned} \tag{9.11}$$

In the last row we have defined the *scaling limit* monodromy matrix $\mathcal{M}(\lambda)$, because we find it again performing the limit (9.1) on (4.25):

$$\begin{aligned}
\mathbf{M}'(\lambda) &= \overleftarrow{\prod_{i=1}^{N/4}} \left[1 + \Delta\mathcal{L}\left(\frac{4i}{N}R, \frac{\mu^{1/2}}{\lambda}\right) + O(\Delta^2)\right]\left[1 + \Delta\mathcal{L}\left(\frac{4i-1}{N}R, \frac{\mu^{1/2}}{\lambda}\right) + O(\Delta^2)\right] \\
&\quad \times \left[1 + \Delta\bar{\mathcal{L}}\left(\frac{4i-2}{N}R, \mu^{1/2}\lambda\right) + O(\Delta^2)\right]\left[1 + \Delta\bar{\mathcal{L}}\left(\frac{4i-3}{N}R, \mu^{1/2}\lambda\right)\right. \\
&\quad \left. + O(\Delta^2)\right] \\
&= \overleftarrow{\prod_{i=1}^{N/4}} \left[1 + \Delta\mathcal{L}\left(\frac{4i}{N}R, \frac{\mu^{1/2}}{\lambda}\right) + \Delta\mathcal{L}\left(\frac{4i-1}{N}R, \frac{\mu^{1/2}}{\lambda}\right)\right. \\
&\quad \left. + \Delta\bar{\mathcal{L}}\left(\frac{4i-2}{N}R, \mu^{1/2}\lambda\right) + \Delta\bar{\mathcal{L}}\left(\frac{4i-3}{N}R, \mu^{1/2}\lambda\right) + O(\Delta^2)\right] \\
&\quad \to \mathcal{P} \exp \frac{1}{2}\int_0^R \mathrm{d}y \left[\bar{\mathcal{L}}\left(y, \frac{\mu^{1/2}}{\lambda}\right) + \mathcal{L}\left(y, \mu^{1/2}\lambda\right)\right] = \mathcal{M}(\lambda).
\end{aligned} \tag{9.12}$$

On the basis of this coincidence we guess the *equivalence* of the theories described by the two off-critical monodromy matrices in the continuum cylinder limit. Combining these heuristic results with the previous ones, we can better support our conjecture according to which the monodromy matrices (4.24) and (4.25) are *equivalent* descriptions of minimal conformal theories perturbed by the primary operator $\Phi_{1,3}$.

10. Similarity with lattice sine–Gordon theory

The interpretation of monodromy matrices $\mathbf{M}$ and $\mathbf{M}'$ as lattice regularized descriptions of $\Phi_{1,3}$ perturbation of CFTs will be reinforced by the results of this section. Indeed, we will show that BEs and transfer matrices eigenvalues, derived for $\mathbf{M}$ and $\mathbf{M}'$, are strictly related to those of lattice sine–Gordon theory (LSGT). In its turn the continuum sine–Gordon theory (SGT) contains the minimal CFTs perturbed by $\Phi_{1,3}$ as a sub-theory derived through quantum group reduction [39].

The continuum SGT on a cylinder is defined by the Hamiltonian

$$H = \int_0^R \mathrm{d}x \left[\frac{1}{2}(\partial_t\Phi)^2 + \frac{1}{2}(\partial_x\Phi)^2 + \frac{m^2}{8\gamma}(1-\cos\sqrt{8\gamma}\Phi)\right] \tag{10.1}$$

where m is the mass parameter and γ is the coupling constant. In [40] the authors found a lattice regularization of the SGT (10.1) and hence they wrote the Bethe equations and the eigenvalues of the transfer matrix. With the definition

$$S \equiv \left(\tfrac{1}{4}m\Delta\right)^2 \tag{10.2}$$

and for $N/4 \in \mathbb{N}$ these can be written as

- Bethe equations

$$\left[\frac{1+S\left(\lambda_s'^2\,\mathrm{e}^{-\mathrm{i}\gamma}+\lambda_s'^{-2}\,\mathrm{e}^{\mathrm{i}\gamma}\right)}{1+S\left(\lambda_s'^{-2}\,\mathrm{e}^{-\mathrm{i}\gamma}+\lambda_s'^{2}\,\mathrm{e}^{\mathrm{i}\gamma}\right)}\right]^{N/4} = \prod_{\substack{r=1\\ r\neq s}}^{l}\frac{\lambda_r'^2\,\mathrm{e}^{-\mathrm{i}\gamma}-\lambda_s'^2\,\mathrm{e}^{\mathrm{i}\gamma}}{\lambda_r'^2\,\mathrm{e}^{\mathrm{i}\gamma}-\lambda_s'^2\,\mathrm{e}^{-\mathrm{i}\gamma}} \tag{10.3}$$

  or, after defining $\lambda_r' = \mathrm{e}^{\alpha_r'}$,

$$\left[\frac{1+2S\cosh\left(2\alpha_s'-\mathrm{i}\gamma\right)}{1+2S\cosh\left(2\alpha_s'+\mathrm{i}\gamma\right)}\right]^{N/4} = \prod_{\substack{r=1\\ r\neq s}}^{l}\frac{\sinh\left(\alpha_s'-\alpha_r'+\mathrm{i}\gamma\right)}{\sinh\left(\alpha_s'-\alpha_r'-\mathrm{i}\gamma\right)}. \tag{10.4}$$

- Eigenvalues of the transfer matrix

$$\begin{aligned}\Lambda^{IK}(\lambda',\{\lambda_r'\}) &= \prod_{r=1}^{l}\frac{\lambda_r'^2\,\mathrm{e}^{\mathrm{i}\gamma}-\lambda'^2\,\mathrm{e}^{-\mathrm{i}\gamma}}{\lambda_r'^2-\lambda'^2}\left[1+S(\lambda'^2\,\mathrm{e}^{-\mathrm{i}\gamma}+\lambda'^{-2}\,\mathrm{e}^{\mathrm{i}\gamma})\right]^{N/4}\\ &\quad+\prod_{r=1}^{l}\frac{\lambda_r'^2\,\mathrm{e}^{-\mathrm{i}\gamma}-\lambda'^2\,\mathrm{e}^{\mathrm{i}\gamma}}{\lambda_r'^2-\lambda'^2}\left[1+S(\lambda'^2\,\mathrm{e}^{\mathrm{i}\gamma}+\lambda'^{-2}\,\mathrm{e}^{-\mathrm{i}\gamma})\right]^{N/4}\end{aligned} \tag{10.5}$$

  or, after defining $\lambda' = \mathrm{e}^{\alpha'}$, $\lambda_r' = \mathrm{e}^{\alpha_r'}$,

$$\begin{aligned}\Lambda^{IK}(\alpha',\{\alpha_r'\}) &= \prod_{r=1}^{l}\frac{\sinh\left(\alpha'-\alpha_r'-\mathrm{i}\gamma\right)}{\sinh\left(\alpha'-\alpha_r'\right)}\left[1+2S\cosh\left(2\alpha'-\mathrm{i}\gamma\right)\right]^{N/4}\\ &\quad+\prod_{r=1}^{l}\frac{\sinh\left(\alpha'-\alpha_r'+\mathrm{i}\gamma\right)}{\sinh\left(\alpha'-\alpha_r'\right)}\left[1+2S\cosh\left(2\alpha'+\mathrm{i}\gamma\right)\right]^{N/4}.\end{aligned} \tag{10.6}$$

If we start from our trigonometric Bethe equations (7.32), (7.38) and eigenvalues (7.34), (7.40) of the two transfer matrices in the off-critical case and make the identifications

$$\beta^2 = \frac{\gamma}{\pi} \qquad \frac{e^{-2\Theta}}{1 + e^{-4\Theta}} = S \qquad \alpha = \alpha' + \frac{i\pi}{2} \qquad \alpha_r = \alpha_r' + \frac{i\pi}{2} \tag{10.7}$$

we then see that our Bethe equations are equal to sine–Gordon ones up to the factors $e^{\mp 2i\pi\beta^2 l}$. And, in addition, our eigenvalues of $\mathbf{T}$ and $\mathbf{T}'$ are proportional, by the factor $e^{\pm\frac{i\pi\beta^2}{2}}\left(\frac{1+e^{-4\Theta}}{4}\right)^{N/4}$, to sine–Gordon eigenvalues (10.6), but the first addend has been multiplied by the factor $e^{\pm i\pi\beta^2 l}$ and the second by the factor $e^{\mp i\pi\beta^2 l}$. The upper sign in the exponentials (*the twist factors*) is for Bethe states diagonalizing $\mathbf{T}$, and the lower sign is for Bethe states diagonalizing $\mathbf{T}'$: the states which diagonalize $\mathbf{T}$ give rise to Bethe equations with twist $e^{-2i\pi\beta^2 l}$, while the states which diagonalize $\mathbf{T}'$ give rise to Bethe equations with twist $e^{+2i\pi\beta^2 l}$.

Twisted versions of Bethe equations and eigenvalues of the transfer matrix for the SGT are already present in the literature. However, usually the twist is introduced *ad hoc* [13], in order to identify the properties of the states under the symmetry of the theory (10.1) $\Phi \to \Phi + \frac{2\pi n}{\sqrt{8\gamma}}$. In contrast, in our case the dynamical twist comes naturally into the theory and, different from the usual approaches to other theories, it depends on the number l of Bethe roots.

For instance, we want to show how we recover the l- and N-independent twist introduced in [13] in the particular case $\beta^2 = \frac{1}{p+1}$, with p a positive integer. We call *the vacuum sector* solutions those sets of Bethe roots corresponding to $l = N/4$ in the limit $N \to \infty$. In this limit, we are obliged to parametrize the chain length as follows (this kind of parametrization has also been used in [38] in the case of the Liouville model)

$$\frac{N}{4} = (p+1)n + \kappa \qquad 0 \leqslant \kappa \leqslant p \qquad n \in \mathbb{N}. \tag{10.8}$$

Indeed, at fixed κ the twist phase factors do not oscillate as $N \to \infty$,

$$e^{\mp 2i\pi\beta^2 l} = e^{\mp 2i\pi\frac{1}{p+1}\frac{N}{4}} \to e^{\mp 2i\pi\frac{1}{p+1}\kappa} \tag{10.9}$$

but become N-independent. Hence, for any κ, the Bethe equations (7.32), (7.38) and the corresponding Bethe state become, in a natural way, respectively, the Bethe equations and the κ-vacuum of the twisted SGT presented in [13]. Besides, for $\kappa \neq 0$ this κ-vacuum is also a state of the pth unitary minimal CFT. This procedure can be repeated also for non-unitary models and for excited states, which are characterized, as well as the vacuum, by their twisting properties. We will come back to this point in a forthcoming paper [29]. Of course, for the non-twisted state ($\kappa = 0$), we obtain the LSGT Bethe equations (10.4) for the vacuum and the corresponding eigenvalue of the transfer matrix proportional to (10.6).

11. Conclusions and perspectives

We have found a generalization of the Yang–Baxter algebra, called the braided Yang–Baxter algebra, as a result of discretization and quantization of the monodromy matrices of two coupled (m)KdV equations. A matrix $Z_{ab}(q)$, independent of the spectral parameter and of the lattice variables, encodes the braiding effect, which is a pure quantum feature and disappears in the classical limit $q \to 1$, because $Z_{ab}(q) \to \mathbf{1}$. By virtue of the commutativity of the braiding matrix Z_{ab} with the quantum R-matrix we have proved that the braided Yang–Baxter algebra still ensures the Liouville integrability, i.e. the transfer matrix commutes for different values of the spectral parameter and therefore generates (an infinite number of) operators in involution. Regarding these operators as a Cartan sub-algebra, a suitable generalization of the algebraic Bethe ansatz technique has been built to construct representations in which they are diagonal. As an effect of the braiding an l-dependent *dynamical* twist appears in the Bethe equations.

We will prove in a forthcoming paper [29] that these representations are vacuum (highest weight) representations for the Hamiltonian operator. In the cylinder continuum limit, we will find non-linear integral equations describing the energy spectrum. The conjecture we have proposed and supported here proved that this spectrum is that of perturbed minimal conformal field theory.

Actually, our left and right (conformal) monodromy matrices (4.22) and (4.23) are in the cylinder continuum limit slightly different from those analysed in [27], and it is very peculiar that they form a braided Yang–Baxter algebra, although those in [27] close a usual Yang–Baxter algebra. Nevertheless, we will see in a forthcoming paper [29] how to build, from our monodromy matrices, others satisfying the unbraided Yang–Baxter relation [41], realizing a deeper link to [27].

In a sequel to [33, 42], one of the authors (DF) in collaboration with M Stanishkov has built a general method of finding hidden symmetries in the classical KdV theory starting from the Lax operator (2.11) of section 2. In particular, a very interesting quasi-local Virasoro algebra has been discovered in [43] and its action on soliton solutions has been studied. Since only some hints have been given about quantization of this intriguing symmetry algebra, it is very interesting to understand how this algebra arises in the quantum context of the present paper.

Eventually, this way of quantizing the simplest KdV theory and of going out of criticality grounds only on algebraic properties of the involved fields/variables and consequently leads very easily to applications to all the generalized KdV theories [41]. Among them the next interesting case would be represented by the quantum $A_2^{(2)}$ KdV depicted in [28], which completes the scenario of integrable perturbations of minimal conformal field theories (i.e. theories without extended conformal symmetry algebra).

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